3

Comparability Graphs, Interval Graphs and Interval Orders

G. R. Brightwell and W. T. Trotter

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3.1 Chapter Overview

We begin this chapter by developing a structural characterization of comparability graphs. Almost all of this material has its origins in the work of Gallai and his classic paper [99]. From this characterization, it is easy to develop a poylnomial time recognition algorithm which will determine whether an input graph is a comparability graph. When the answer is yes, we will provide a transitive orientation, and when the answer is no, we will be able to explain why. Our treatment of comparability graphs will include a number of elementary propositions, and in some cases, we provide only a sketch of the proof—although we encourage the reader to fill in the details, as the subject area has trapped many an unsuspecting explorer.

A surprisingly large percentage of combinatorial problems for partially ordered sets deals with the special class of interval orders. Building on the characterization of comparability graphs, we develop in this chapter the key structural properties of interval orders which will be used in several of the following chapters.

3.2 A Structural Characterization of Comparability Graphs

We consider a *transitive orientation* T of a comparability graph G as a strict partial order T on the vertex set of G so that (1) for every edges xy, exactly one of (x, y) and (y, x) belongs to T; and (2) if $(x, y), (y, z) \in T$, then xz is an edge in G and $(x, z) \in T$.

We start with the following elementary proposition.

Proposition 3.2.1 *Let G be a comparability graph, and let T be a transitive orientation of G. Let x, y and z be vertices in G. If xy and yz are edges in*

G but yz is not an edge, then exactly one of the following two statements is true:

(i) $(x, y) \in T$ *and* $(x, z) \in T$.
(ii) $(y, x) \in T$ *and* $(z, x) \in T$.

It is important to note that the requirement that y not be adjacent to z is satisfied when $y = z$.

3.2.1 Comparability Testing Graphs

When G is a graph, we define the *comparability testing graph* of G, denoted $\mathrm{CT}(G)$ as follows. The vertex set of $\mathrm{CT}(G)$ is $\{(x, y) : xy \text{ is an edge in } G\}$. The edges in $\mathrm{CT}(G)$ are determined by the following two rules:

(i) If (x, y) and (x, z) are distinct vertices of $\mathrm{CT}(G)$, with y not adjacent to z in G, then (x, y) is adjacent to (x, z) in $\mathrm{CT}(G)$.
(ii) If (y, x) and (z, x) are distinct vertices of $\mathrm{CT}(G)$, with y not adjacent to z in G, then (y, x) is adjacent to (z, x) in $\mathrm{CT}(G)$.

The following result is immediate.

Proposition 3.2.2 *Let G be a comparability graph, and let C be a component of* $\mathrm{CT}(G)$. *If T is a transitive orientation of G, then exactly one of the following two statements is true:*

(i) $C \subseteq T$.
(ii) $C \cap T = \emptyset$.

We say that a graph G is *consistent* if there there is no vertex (x, y) in $\mathrm{CT}(G)$ so that (x, y) and (y, x) belong to the same component of $\mathrm{CT}(G)$. It is clear that a comparability graph is consistent. As we shall see, the converse is also true—although we will need some additional background material before presenting the proof of the following theorem.

Theorem 3.2.3 *A graph is a comparability graph if and only if it is consistent.*

Before closing this subsection, we note the following elementary fact.

Proposition 3.2.4 *An induced subgraph of a consistent graph is also consistent.*

3.2.2 Circuits in Graphs

Other researchers have elected to state Theorem 3.2.3 in a slightly different manner. Let t be an integer with $t \geq 3$. A sequence $\sigma = (x_1, x_2, \ldots, x_t)$ of vertices in a graph G is called a *circuit* when $x_i x_{i+1}$ is an edge in G for each $i = 0, 1, \ldots, t$. Of course, when $i = t$, we simply mean that $x_t x_0$ is an edge in G. On the other hand, we do not require that the vertices in a circuit be distinct. In particular, it is always possible for x_i to be the same vertex as x_{i+2}.

A *triangular chord* in a circuit $\sigma = (x_1, x_2, \ldots, x_t)$ is a distinct pair $x_i x_{i+2}$ which is an edge in G.

With this terminology, Theorem 3.2.3 can be restated as follows—and this is the statement you will find in the papers of Ghoulá-Houri [99] and Gilmore and Hoffman [99].

Theorem 3.2.5 *A graph G is a comparability graph if and only if every odd circuit of length at least 5 has a triangular chord.*

3.2.3 Partitive Sets in Graphs

If G is a graph, a set S of vertices is said to be *partitive* in G if for every vertex not in S, exactly one of the following two statements holds:

(i) x is adjacent to s, for every $s \in S$.
(ii) x is not adjacent to s, for every $s \in S$.

A single vertex is always partitive, as is the set of all vertices in G; these are called *trivial* partitive sets. All other partitive sets are *non-trivial.*

In the discussion which follows, when $G = (V, E)$ is a graph and D is a set of vertices in the comparability testing graph of G, we let $E(D) = \{uv \in E : (u, v) \in D\}$. Also, we let $S(D)$ denote the subset of V consisting of all vertices which are endpoints of one or more edges in $E(D)$.

Proposition 3.2.6 *Let G be a graph, let $\mathrm{CT}(G)$ be its comparability testing graph and let C be a component of $\mathrm{CT}(G)$. Then*

(i) *$S(C)$ is a partitive set in G.*
(ii) *$\big(S(C), E(C)\big)$ is a connected subgraph of G.*

3.2.4 A Reduction Lemma

The following lemma explains why partitive sets are useful in characterizing comparability graphs.

Lemma 3.2.7 *Let $G = (V, E)$ be a graph, let S be a partitive set in G, and let s_0 be any vertex in S. Then let G_1 be the subgraph induced by S and let G_2 be the subgraph induced by $(V - S) \cup \{s_0\}$. Then*

(i) *G is a comparability graph if and only if both G_1 and G_2 are comparability graphs.*
(ii) *G is consistent if and only if both G_1 and G_2 are consistent.*

Proof We prove the first statement, and note that the second follows along similar lines.

An induced subgraph of a comparability graph is also a comparability graph. So if G is a comparability graph, S is a partitive set in G, and $s_0 \in S$, then G_1 and G_2 are induced subgraphs of G and are therefore comparability graphs.

Now suppose that S is a partitive set in G, $s_0 \in S$, and G_1 and G_2 are comparability graphs. Let T_1 and T_2 be transitive orientations of G_1 and G_2 respectively. Construct a transitive orientation T of G by setting

(i) If $x, y \in S$, then $(x, y) \in T$ if and only if $(x, y) \in T_1$.
(ii) If $x, y \notin S$, then $(x, y) \in T$ if and only if $(x, y) \in T_2$.
(iii) If $x \in S$, $y \notin S$, then (1) $(x, y) \in T$ if and only if $(s_0, y) \in T_2$ and $(y, x) \in T$ if and only $(y, s_0) \in T_2$.

□

3.2.5 Completing the Proof

Now we return to Theorem 3.2.3 and show that if $G = (V, E)$ is consistent, then it is a comparability graph. We proceed by induction on the number of vertices in G. By inspection, all graphs on at most 4 vertices are comparability graphs, so we will assume that the theorem holds whenever G has at most k vertices, where $k \geq 4$. We then consider the case when G has $k + 1$ vertices. If G has a non-trivial partitive set S, then the result follows from Lemma 3.2.7. So we will assume that G has no non-trivial partitive sets. Note that this implies that G is connected.

Let C be an arbitrary component of the comparability testing graph $\mathrm{CT}(G)$. To complete the proof, we will show that C is a transitive orientation of G.

Claim 3.2.8 *C is an irreflexive and transitive binary relation.*

Proof The fact that C is irreflexive follows from the assumption that G is consistent. We now show that it is transitive. We argue by contradiction. Of all triples (a, b, c) for which $(a, b), (b, c) \in C$ and $(a, c) \notin C$, choose one for which the distance from (a, b) to (b, c) in $\mathrm{CT}(G)$ is as small as possible. Then let $(p_1, p_2, \ldots, p_t)$ be a shortest path from (a, b) to (b, c) in $\mathrm{CT}(G)$.

First note that $a \neq c$, for if $a = c$, then C contains both (a, b) and $(b, c) = (b, a)$, which is a contradiction. Second, note that ac is an edge in G, for if a is not adjacent to c in G, then (a, b) is adjacent to (c, b) in $\mathrm{CT}(G)$, which implies that (b, c) and (c, b) belong to C.

For each $i = 1, 2, \ldots, t$, let $p_i = (x_i, y_i)$. Of course, $x_1 = a$, $y_1 = x_t = b$, and $y_t = c$. Then let j be the least positive integer for which $x_1 = x_2 = \cdots = x_j = a$ and $x_{j+1} \neq a$. Then $(a, y_i) \in C$ for each $i = 1, 2, \ldots, j$ and y_i is not adjacent to y_{i+1} for all $i = 1, 2, \ldots, j-1$. Also $y_{j+1} = y_j$; and x_{j+1} is not adjacent to a in G.

Observe that $c \notin \{y_1, y_2, \ldots, y_j\}$ and that $(y_1, c) = (b, c) \in C$. If for some i with $2 \leq i \leq j$, y_i is not adjacent to c in G, then (a, c) is adjacent to (a, y_i) in $\mathrm{CT}(G)$, so $(a, c) \in C$. The contradiction allows us to conclude that $cy_i \in E$, for all $i = 1, 2, \ldots, j$. Also, since $(y_1, c) \in C$, we conclude that $(y_i, c) \in C$ for every $i = 1, 2, \ldots, j$.

It follows that

$$\big(p_{j+1}, p_{j+2}, \ldots, p_{t-1}, p_t, (y_1, c), (y_2, c), \ldots, (y_j, c)\big)$$

is a path from $p_{j+1} = (x_{j+1}, y_{j+1}) = (x_{j+1}, y_j)$ to (y_j, c). However, this path consists of $t-1$ vertices, and thus $(x_{j+1}, c) \in C$. Since x_{j+1} is not adjacent to a, it follows that (a, c) is adjacent to (x_{j+1}, c) in $\mathrm{CT}(G)$. Thus $(a, c) \in C$. $\square$

Claim 3.2.9 $E = E(C)$, *for every component* C *of* $\mathrm{CT}(G)$.

Proof Suppose the claim is false. Then let C be a component of $\mathrm{CT}(G)$ for which $E(C) \subsetneq E$. Let $H = (S(C), E(C))$. Then $|S(C)| \geq 2$. Furthermore, from Proposition 3.2.6, we know that $S(C)$ is partitive. Therefore $S(C) = V$.

Since H is connected, it follows that for every edge $e \in E - E(C)$, there is a path in H from one end of e to the other.

Of all edges in $E - E(C)$, choose an edge $e = xy$ so that the length of the path in H from x to y is as short as possible. We claim that this path consists of three vertices. Suppose to the contrary that the path is $(x = u_0, u_1, \ldots, u_t = y)$ with $t \geq 3$. If u_1 is not adjacent to y in G, then (x, y) is adjacent to (u_0, u_1) in $\mathrm{CT}(G)$ so $e \in E(C)$. So we conclude that $f = u_1 y \in E$. If $f \in E(C)$, then (u_0, u_1, y) is a path from x to y in H. If

$f \notin E(C)$, then $(u_1, u_2, \ldots, u_t)$ is a shorter path in H from one end point of f to the other, contradicting our choice of e.

It follows that either (1) $(x, u_1) \in C$ and $(y, u_1) \in C$, or (2) $(u_1, x) \in C$ and $(u_1, y) \in C$. Without loss of generality, we assume that the first statement applies. Let C' denote the component of $\mathrm{CT}(G)$ containing the vertex (x, y) and let $H' = (V, E(C'))$. Then H' is also a connected graph. Note that $E(C) \cap E(C') = \emptyset$.

Consider all subsets $S \subseteq V - \{u_1\}$ for which the following two conditions are satisfied.

(i) $(s, u_1) \in C$ for every $s \in S$.
(ii) S induces a connected subgraph of H'.

Of all such sets, choose one with $|S|$ as large as possible. Since S is not an autonomous set in G, there exists a vertex $y \in V - S$ so that y is adjacent in G to some but not all vertices of S. Then $y \neq u_1$. Since S induces a connected subgraph of H', it follows that there are distinct vertices $s_1, s_2 \in S$ so that $f = s_1 s_2$ is an edge from $E(C')$, $ys_1 \in E$ and $ys_2 \notin E$. Then (y, s_1) is adjacent to (s_1, s_2) in $\mathrm{CT}(G)$ so $ys_1 \in E(C')$. Therefore $S' = S \cup \{y\}$ induces a connected subgraph of H'.

If y is not adjacent to u_1, then (s_1, u_1) is adjacent to (s_1, y) in $\mathrm{CT}(G)$ and thus $s_1 y \in E(C) \cap E(C')$, which is a contradiction. Since s_2 is not adjacent to y in G, we know (s_2, u_1) is adjacent to (y, u_1) in $\mathrm{CT}(G)$. Thus S' satisfies both conditions and we have a contradiction. $\Box$

It follows from the preceding claim that $\mathrm{CT}(G)$ has exactly two components each of which is a transitive orientation of G. This completes the proof of the theorem.

3.2.6 The Family of Transitive Orientations

Let $\mathbf{P} = (X, P)$ be a poset and let S be a proper subset of X. We say S is *autonomous* in $\mathbf{P}$ if for every $x \in X - S$, exactly one of the following three statements holds:

(i) $x < s$ for every $s \in S$.
(ii) $x > s$ for every $s \in S$.
(iii) $x \| s$ for every $s \in S$.

When S is autonomous in $\mathbf{P}$, note that S is partitive in the comparability graph of $\mathbf{P}$. However, the converse is not true.

When S is autonomous in $\mathbf{P}$, we we can form a new partial order Q on X as follows.

(i) If $x, y \in X - S$, set $x < y$ in Q if and only if $x < y$ in P.
(ii) If $x \in X - S$ and $s \in S$, set $x < s$ in Q if and only if $x < s$ in P and $x > s$ in Q if and only if $x > s$ in P.
(iii) If $x, y \in S$, set $x < y$ in Q if and only if $x > y$ in P.

The partial order Q is said to have been obtained from P by *reversing* the autonomous set S. Note that (X, P) and (X, Q) have the same comparability graph.

The following fundamentally important result, also due to Gallai [99], admits an easy inductive proof. It is the basis of many of the "comparability invariant" proofs which follow in subsequent chapters.

Theorem 3.2.10 *Let P and Q be partial orders on a set X so that the posets $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (X, Q)$ have the same comparability graph G. Then there exists a sequence $P_0, P_1, \ldots, P_t$ of partial orders on X and a sequence $S_0, S_1, \ldots, S_{t-1}$ of subsets of X so that $P = P_0$, $Q = P_t$ and for each $i = 0, 1, \ldots, t-1$, S_i is an autonomous set in P_i, and P_{i+1} is obtained from P_i by reversing S_i.*

Proof Let P and Q be distinct strict partial orders from $\mathcal{T}$. Set $P_0 = P$. If P_i has been defined for some $i \geq 0$ and $P_i \neq Q$, choose $(x, y) \in P_i - Q$. Then let C be the component of $\mathrm{CT}(G)$ which contains the vertex (x, y). Then $C \subset P_i - Q$. Furthermore $S(C)$ is an autonomous set in P_i. Then let P_{i+1} be obtained from P_i by reversing the automous set $S(C)$. Furthermore, $C \subset P_{i+1} \cap Q$. □

3.2.7 Algorithmic Issues

Let G be a graph on n vertices. It is easy to see that the structural characterization of comparability graphs developed above allows for the development of an algorithm, whose running time will be polynomial in n, for testing whether a graph is a comparability graph. Note that the comparability testing graph $\mathrm{CT}(G)$ has at most $2n(n-1)$ vertices. So a spanning tree algorithm will determine its components, and from this information, the issue of whether G is consistent can be quickly settled.

When the input graph G is not consistent, we find an edge $xy \in E$ for which both (x, y) and (y, x) belong to the same component of $\mathrm{CT}(G)$. A path from (x, y) to (y, x) in $\mathrm{CT}(G)$ is then a "certificate" that G is not a comparability graph.

On the other hand, if G is consistent, then the components of $\mathrm{CT}(G)$

come in pairs, one being the reverse of the other. A transitive orientation is obtained by taking one of the two, for each such pair.

3.2.8 Characterizing Comparability Graphs by Forbidden Subgraphs

Since any induced subgraph of a comparability graph is also a comparability graph, it follows that there is a minimum family $\mathcal{C}$ of graphs so that a graph G is a comparability graph if and only if it does not contain a graph from $\mathcal{C}$ as an induced subgraph. In other words, the family $\mathcal{C}$ provides a *forbidden subgraph* characterization of comparability graphs.

Despite the fact that recognizing comparability graphs is easy, it is surprisingly difficult to find this minimum family $\mathcal{C}$. This formidable task was accomplished by Gallai in 1967xx, and his classic paper [99] is a masterful piece of combinatorial mathematics, worthy of careful study.

It is relatively straightforward (although time consuming) to show that the graphs displayed in Figure 3.2.8 belong to $\mathcal{C}$. Also, the *complements* of the graphs shown in Figure 3.2.8 belong to $\mathcal{C}$. Gallai succeeded in showing that these are the *only* graphs in $\mathcal{C}$. We know of no easy proof for this result. In fact, in retrospective, Gallai's relatively lengthy argument can only be viewed as elegant and insightful.

Theorem 3.2.11 (Gallai) *The minimum list $\mathcal{C}$ of forbidden graphs for comparability graphs consists of the graphs in Figure 3.2.8 and the complements of the graphs in Figure 3.2.8.*

3.3 Interval Orders and Interval Graphs

A poset $\mathbf{P} = (X, P)$ is called an *interval order* if there is a function I assigning to each element $x \in X$ a closed interval $I(x) = [a_x, b_x]$ of the real line $\mathbb{R}$ so that for all x, $y \in X$, $x < y$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. We call I an *interval representation* of $\mathbf{P}$, or just a *representation* for short. For brevity, whenever we say that I is a representation of an interval order $\mathbf{P} = (X, P)$, we will use the alternate notation $[a_x, b_x]$ for the closed interval $I(x)$. Also, we let $|I(x)|$ denote the *length* of the interval, i.e., $|I(x)| = b_x - a_x$.

Note that end points of intervals used in a representation need not be distinct. In fact, distinct points x and y from X may satisfy $I(x) = I(y)$. We even allow degenerate intervals. On the other hand, a representation is said to be *distinguishing* if all intervals are non-degenerate and all end points

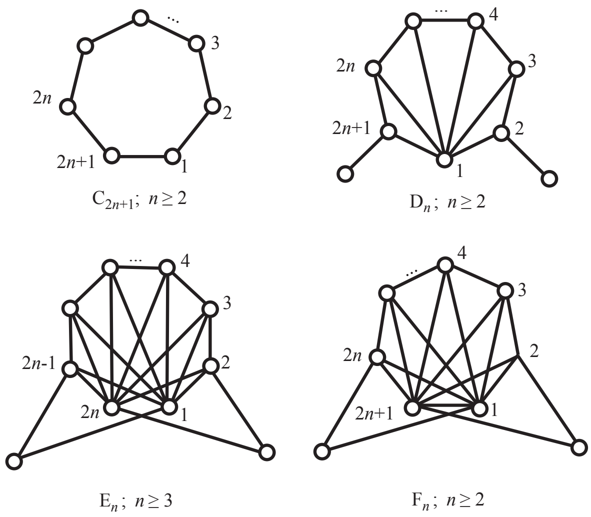

Fig. 3.1. Comparability Graphs - Part 1

are distinct. It is easy to see that every interval order has a distinguishing representation. In fact, since we are concerned only with finite posets, we could have just as well required that all intervals used in the representation be open.

Analogously, a graph $\mathbf{G} = (V, E)$ is an *interval graph* when there is a function I which assigns to each vertex $x \in V$ a closed interval $I(x) = [a_x, b_x]$ of $\mathbb{R}$ so that $\{x, y\} \in E$ if and only if $I(x) \cap I(y) \neq \emptyset$. As before, we call I an *interval representation* of G and note that, if desired, we may assume I is distinguishing.

Throughout this monograph, we will move back and forth between posets and graphs in discussions about a family of intervals. The interval graph determined by a family of intervals is just the incomparability graph of the interval order. Chains correspond to independent sets and antichains correspond to cliques.

3.3.1 Classical Representation Theorems

A good fraction of the early research on interval graphs and interval orders was focused on characterization issues. Recall that a graph is *chordal* (some researchers prefer to say the graph is *triangulated*) if it does not contain a cycle on four or more vertices as an induced subgraph. Also, a vertex x in a

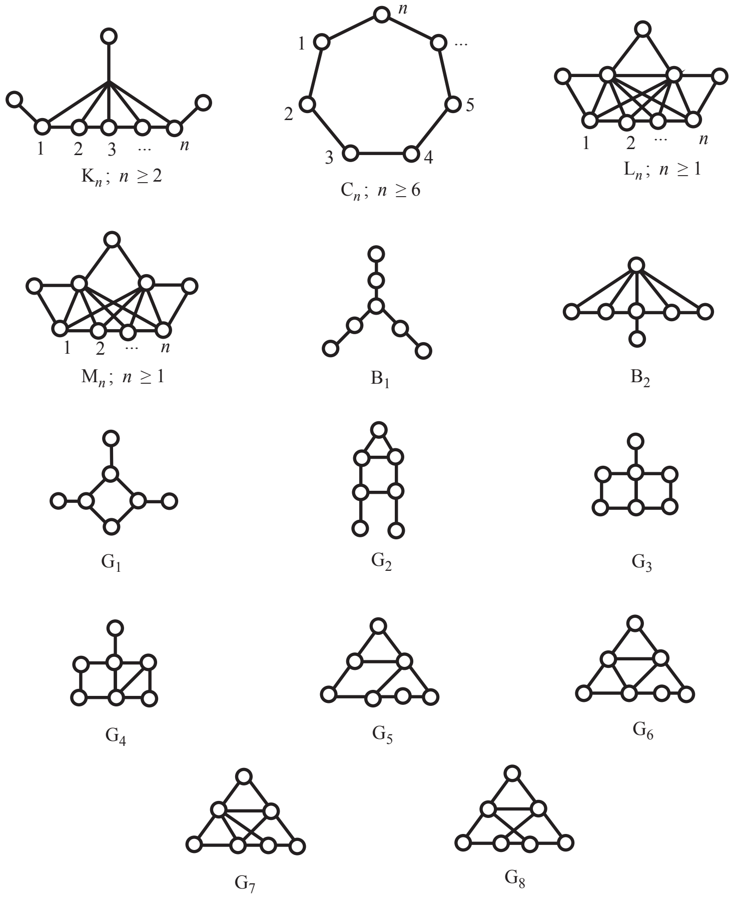

Fig. 3.2. Comparability Graphs - Part 2

graph G is *simplicial* if its neighborhood is a complete subgraph of G. Here is an elementary proposition linking these two concepts.

Proposition 3.3.1 *A graph G is chordal if and only if every induced subgraph has a simplicial vertex.*

Chordal graphs are a well-studied class of perfect graphs. The issue of perfection is settled by the following elementary result.

Proposition 3.3.2 *Let G be a chordal graph on n vertices and let $L = (x_1, x_2, \ldots, x_n)$ be a listing of the vertices of G so that for each $i = 1, 2, \ldots, n$, vertex x_i is a simplicial vertex in the induced subgraph determined by $\{x_1, x_2, \ldots, x_i\}$.*

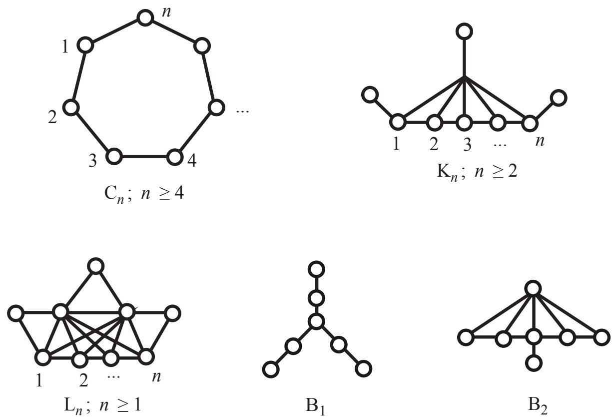

Fig. 3.3. Forbidden Subgraphs for Interval Graphs

Then the First Fit coloring algorithm colors G with $\omega(G)$ colors, where $\omega(G)$ is the maximum clique size of G.

Obviously, interval graphs are chorday, but it is natural to ask whether all chordal graphs are interval graphs. This is not true. In fact, not all trees are interval graphs, e.g., the subdivision of $K(1,3)$ is not an interval graph.

Three distinct vertices x, y and z in a graph G are said to form an *asteroidal triple* when for each two vertices in $\{x,y,z\}$, there is a path joining them, with no vertex on the path adjacent to the third. For example, the three leaves in a subdivision of $K(1,3)$ form an asteroidal triple.

Theorem 3.3.3 (Lekkerkerker and Boland) *A chordal graph is an interval graph if and only if it does not contain any asteroidal triples.*

Boland and Lekkerkerker [99] used the preceding theorem to develop a forbidden subgraph characterization of interval graphs; the argument is quite complicated, although as we will see a bit later, a very simple proof can be derived using Gallai's characterization of comparability graphs.

Theorem 3.3.4 (Lekkerkerker and Boland) *The minimum list $\mathcal{I}$ of forbidden graphs for interval graphs consists of the graphs shown in Figure 3.3.1.*

3.3.2 Weak Orders

A finite poset $\mathbf{P} = (X, P)$ is called a *weak order* if there exists a function $f : X \to \mathbb{R}$ so that for all $x, y \in X$ with $x \neq y$,

(i) $x < y$ in P if and only if $f(x) < f(y)$ in $\mathbb{R}$, and
(ii) $x \| y$ if and only if $f(x) = f(y)$.

The following elementary result is one of our exercises.

Proposition 3.3.5 *Let* $\mathbf{P} = (X, P)$ *be a poset. Then the following are equivalent.*

(i) $\mathbf{P}$ *is a weak order.*
(ii) $\mathbf{P}$ *does not contain* $\mathbf{2} + \mathbf{1}$ *as a subposet.*
(iii) $\mathbf{P}$ *is the lexicographic sum of a family of antichains over a chain.*

Given a representation I of an interval order $\mathbf{P} = (X, P)$, there are two natural weak orders defined on X by the end points. The *ordering by left end points* L defined by $x < y$ in L if and only if $a_x < a_y$ in $\mathbb{R}$ and the *ordering by right end points* R defined by $x < y$ in R if and only if $b_x < b_y$ in $\mathbb{R}$. When the representation is distinguishing, these weak orders are linear orders.

3.3.3 Characterizing Interval Orders

Interval orders admit a very simple and elegant characterization. The following more comprehensive result summarizes the work of Fishburn [99], Greenough [99] and Bogart-SombodyElse [99].

Theorem 3.3.6 *Let* $\mathbf{P} = (X, P)$ *be a poset. Then the following are equivalent.*

(i) $\mathbf{P}$ *is an interval order.*
(ii) $\mathbf{P}$ *does not contain* $\mathbf{2} + \mathbf{2}$ *as a subposet.*
(iii) *Whenever* $x < y$ *and* $z < w$ *in* P*, then either* $x < w$ *or* $z < y$ *in* P.
(iv) *For every* $x, y \in X$*, either* $D(x) \subseteq D(y)$ *or* $D(y) \subseteq D(x)$.
(v) *For every* $x, y \in X$*, either* $U(x) \subseteq U(y)$ *or* $U(y) \subseteq U(x)$.

Proof The equivalence of the last four statements is immediate. We now show that statement 1 is equivalent to 4. Suppose first that $\mathbf{P} = (X, P)$ is an interval order and that I is an interval representation of $\mathbf{P}$. Let $x, y \in X$; without loss of generality, we may assume $a_x \leq a_y$ in $\mathbb{R}$. Then $D(x) \subseteq D(y)$.

Now suppose that statement 4 holds for a poset $\mathbf{P} = (X, P)$. We show

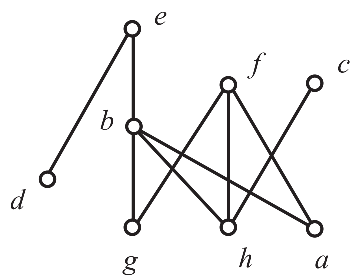

Fig. 3.4. An Interval Order

that $\mathbf{P}$ is an interval order. Let $Y = \{D(x) : x \in X\}$, and let $m = |Y|$. Then define a linear order L on Y by $D < D'$ in L if $D \subsetneq D'$. Then label the sets in Y so that $D_1 < D_2 < \cdots < D_m$ in L. For each $x \in X$, let $F(x) = [i, j]$, where $D(x) = D_i$ and $j = m$ if x is maximal, and $D_{j+1} = \cap\{D(y) : x < y$ in $P\}$, otherwise. $\square$

One advantage to the proof given here for Fishburn's representation theorem for interval orders is that the total number of end points used in the representation is minimal. Also, note that we have the following not entirely trivial proposition, first noticed by Greenough.

Proposition 3.3.7 *Let $\mathbf{P} = (X, P)$ be an interval order. Then $|\{D(x) : x \in X\}| = |\{U(x) : x \in X\}|$.*

Example 3.3.8 *The poset P shown in Figure 3.3.3 is an interval order. In the table below, we list the up sets and down sets for each of the elements in P, noting that there are four distinct down sets which can be labeled as D_1, D_2, D_3 and D_4 so that $D_1 \subsetneq D_2 \subsetneq D_3 \subsetneq D_4$.*

Similarly, there are four distinct up sets and we label them as U_1, U_2, U_3 and U_4 so that $U_4 \subsetneq U_3 \subsetneq U_2 \subsetneq U_1$. The bold face numbers correspond to these labelings.

Finally, in the right most column, we associate with each element of P the interval determined by the assignment of the bold face numbers in the first two columns

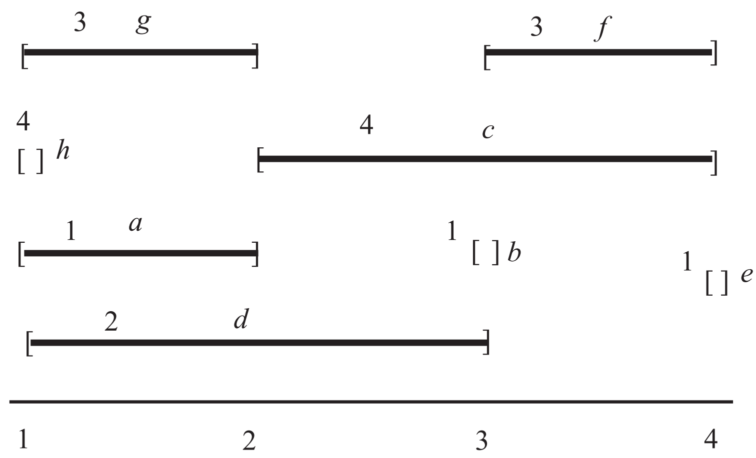

Fig. 3.5. First Fit Coloring - Left Endpoint Order

$D(a) = \emptyset$ ***1***	$U(a) = \{b, e, f\}$ ***2***	$I(a) = [1, 2]$
$D(b) = \{a, g, h\}$ ***3***	$U(b) = \{e\}$ ***3***	$I(b) = [3, 3]$
$D(c) = \{h\}$ ***2***	$U(c) = \emptyset$ ***4***	$I(c) = [2, 4]$
$D(d) = \emptyset$ ***1***	$U(d) = \{e\}$ ***3***	$I(d) = [1, 3]$
$D(e) = \{a, b, d, g, h\}$ ***4***	$U(e) = \emptyset$ ***4***	$I(e) = [4, 4]$
$D(f) = \{a, g, h\}$ ***3***	$U(f) = \emptyset$ ***4***	$I(f) = [3, 4]$
$D(g) = \emptyset$ ***1***	$U(g) = \{b, e, f\}$ ***2***	$I(g) = [1, 2]$
$D(h) = \emptyset$ ***1***	$U(h) = \{b, c, e, f\}$ ***1***	$I(h) = [1, 1]$

In Figure 3.3.8, we show the resulting representation. On this same figure, we illustrate how First Fit will partition the interval order into chains—when applied using the ordering by left endpoints (how ties is broken doesn't matter). This is just the ordering for chordal graphs we discussed in the preceding section. Note that the width of P *is* 4 *because the highest color used is* 4*. There are three maximum antichains:* $\{a, d, g, h\}$, $\{a, c, d, g\}$ *and* $\{b, c, d, f\}$*. The coloring algorithm will find the first two of these using the left endpoint of intervals receiving the highest color. The third maximum antichain could be found by a linear scan.*

3.3.4 Semiorders

An interval order $\mathbf{P} = (X, P)$ is called a *semiorder* if there is a constant c for which it has an interval representation F such that the length of the interval $F(x)$ is exactly c, for every $x \in X$. From a modern perspective, it would perhaps be more natural if these objects were called *constant length* interval orders or just *unit interval orders.* But we are now quite stuck with the term: semiorders.

For semiorders, we have the following representation theorem, the principal part of which is due to Scott and Suppes [99].

Theorem 3.3.9 *Let* $\mathbf{P} = (X, P)$ *be an interval order. Then the following statements are equivalent.*

(i) $\mathbf{P}$ *is a semiorder.*
(ii) $\mathbf{P}$ *does not contain* $\mathbf{3} + \mathbf{1}$ *as a subposet.*
(iii) *Whenever* $x < y < z$ *and* $w \| y$ *in* P*, then either* $x < w$ *or* $w < z$ *in* P*.*
(iv) *The binary relation* $W = \big\{(x, y) \in X \times X : x = y\big\} \cup \big\{(x, y) \in X \times X : D(x) \subseteq D(y), U(y) \subsetneq U(x)\big\} \cup \big\{(x, y) \in X \times X : D(x) \subsetneq D(y), U(y) \subseteq U(x)\big\}$ *is a weak order on* X*.*

Proof The equivalence of statements 2, 3 and 4 is immediate. We show that statements 1 and 4 are equivalent. First let $\mathbf{P} = (X, P)$ be a semiorder and let I be an interval representation in which all intervals have length c. Let $I(x) = [a_x, a_x + c]$, for every $x \in X$. Then $(x, y) \in W$ if and only if $a_x \le a_y$ in $\mathbb{R}$, so that W is a weak order on X.

Now suppose that statement 4 holds for a poset $\mathbf{P} = (X, P)$. We show that $\mathbf{P}$ is a semiorder. We actually prove something stronger. Let L be any linear order on X extending the weak order W. Proceeding by induction on $|X|$, we show that there exists a distinguishing interval representation I of $\mathbf{P}$ which assigns to each $x \in X$ a unit length interval $I(x) = [a_x, a_x + 1]$ such that for all $x, y \in X$, $a_x < a_y$ in $\mathbb{R}$ if and only if $x < y$ in L.

Noting that the claim holds trivially when $|X| = 1$, consider the inductive step. Suppose that L orders X as $x_1 < x_2 < \ldots, < x_n$. Let $Y = X - \{x_n\}$, let $Q = P(Y)$ and $L' = L(Y)$. In the poset $\mathbf{Q} = (Y, Q)$, let W'' be the binary relation defined in statement 4 for the subposet $\mathbf{Q}$. Then let $W' = W(Y)$. Then $W'' \subseteq W' \subseteq L'$.

It follows that $\mathbf{Q}$ is a semiorder and that there exists a distinguishing representation I' of $\mathbf{Q}$ so that for all $y, z \in Y$, $a_y < a_z$ if and only if $y < z$ in L'. Also, $y < z$ in Q if and only if $a_y + 1 < a_z$. We now show that this representation can be extended by an appropriate choice of an interval $I(x_0) = [a_{x_n}, a_{x_n} + 1]$ for x_0. If $y < x_n$ for every $y \in Y$, let $a = \max\{a_y : y \in Y\}$ and set $a_{x_n} = 2 + a$.

So we may assume that $S = \{y \in Y : y \| x_n\} \neq \emptyset$. It follows that there is a positive integer i so that $S = \{x_i, x_{i+1}, \ldots, x_{n-1}\}$, and $[a, b] = \cap\{I(y) : y \in S\}$ is a nondegenerate interval. If $D(x_n) = \emptyset$, set $a' = a$; otherwise, set

$a' = \max\{a_y + 1 : y < x_n\}$. In either case, note that $a' < b$ in $\mathbb{R}$. It follows that we may take a_n as any real number between a' and b distinct from any end point previously chosen. $\square$

There is an important corollary to the Scott-Suppes theorem for semiorders. An interval order $\mathbf{P} = (X, P)$ is said to be *proper* if it admits an interval representation I so that if $x, y \in X$ and $x \neq y$, then $I(x) \not\subseteq I(y)$ and $I(y) \not\subseteq I(x)$. A semiorder is obviously proper but from the preceding theorem, it follows that a proper interval order is also a semiorder.

3.3.5 Semiorders and Catalan Numbers

Here is a little combinatorial gem for semiorders. The result is widely credited to Dean and Keller [99], but it was actually known previously to SomebodyElse [99].

Theorem 3.3.10 *For each $n \geq 1$, the number of unlabeled semiorders on n points is the Catalan number $\binom{2n-2}{n-1}/n$.*

Proof Let I be a distinguishing interval representation of the semiorder P and let L be the linear order on the ground set determined by the left end points. Then the incidence (0–1) matrix M defined by $m(i,j) = 1$ if $i < j$ in P, and $m(i,j) = 0$ otherwise is a Catalan walk. $\square$

3.4 Further Remarks on Characterization Issues

Clearly, there are polynomial time algorithms for testing whether a poset is an interval order, a semiorder, or a weak order—just because in each case, membership is determined by excluding a finite number of forbidden posets. Now consider the problem of testing whether a graph G is an interval graph. First test the complement of G to see whether it is a comparability graph. If not, then G is not an interval graph. If yes, then let P be any partial order of the vertex set V of G so that the comparability graph of (X, P) is the complement of G, noting that G is an interval graph if and only if the answer is yes.

Furthermore, using Fishburn's theorem, it is easy to derive Boland and Lekkerkerker's forbidden subgraph characterization of interval graphs from Gallai's work on comparability graphs. Note that a graph G belongs to the list characterizing interval graphs if and only if the complement belongs to $\mathcal{C}$ and does not contain a four cycle.

The best known algorithm for testing whether a graph is an interval graph is due to Booth and Lueker [99] and has running time $O(n^2)$. Their algorithm uses a special kind of data structure, called a PQ–tree.

3.5 Stuff Left to Do

1. Bibliography hasn't even been started.
2. Historical notes. Nothing done here.
3. Chapter needs to be read carefully for consistency as others have underone significant revision.