Order **15:** 167–182, 1999.

Geometric Containment Orders: A Survey

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(Received: 13 May 1998; accepted: 30 December 1998)

Abstract. A partially ordered set $(X, \prec)$ is a geometric containment order of a particular type if there is a mapping from X into similarly shaped objects in a finite-dimensional Euclidean space that preserves $\prec$ by proper inclusion. This survey describes most of what is presently known about geometric containment orders. Highlighted shapes include angular regions, convex polygons and circles in the plane, and spheres of all dimensions. Containment orders are also related to incidence orders for vertices, edges and faces of graphs, hypergraphs, planar graphs and convex polytopes. Three measures of poset complexity are featured: order dimension, crossing number, and degrees of freedom.

Mathematics Subject Classification (1991): 06A10.

Key words: degrees of freedom, dimension, inclusion order.

1. Introduction

In their seminal paper on dimensions of partial orders, Dushnik and Miller (1941) observed that for every poset $P = (X, \prec)$ there is a family $\mathcal{S}$ of subsets of a given set S and a mapping f from X into $\mathcal{S}$ such that

$$\forall x, y \in X, \quad x \prec y \Leftrightarrow f(x) \subset f(y). \tag{1.1}$$

When this holds for some $f\colon X \to \mathcal{S}$, we say that P is *$\mathcal{S}$-representable*. Dushnik and Miller also proved for order dimension that $\dim(P) \leq 2$ if and only if $\mathcal{S}$ can be chosen as a family of intervals in a linearly ordered set. A restricted-cardinality version of their result for order dimension at most 2 is

THEOREM 1. *Suppose $P = (X, \prec)$ is a poset for which X is countable. Then $\dim(P) \leq 2$ if and only if P is $\mathcal{S}$-representable when $\mathcal{S}$ is the family of closed and bounded intervals in $\mathbb{R}$.*

Theorem 1 qualifies as the first significant result in the theory of geometric containment orders. Apart from questions of cardinality and end-point restrictions, it is

* The research of the second author was supported by the Office of Naval Research (U.S.) and the Deutsche Forschungsgemeinschaft.

the preeminent geometric containment theorem for $\mathbb{R}$. In $\mathbb{R}^2$, a host of interesting new possibilities arise for $\mathcal{S}$, including circular disks, convex polygons, regular n-gons similarly oriented, and angular wedges. Attractive candidates for higher-dimensional Euclidean spaces are spheres, polyhedra, boxes, and translations of cones.

Our aim is to describe much of what is presently known about geometric containment orders. As a working definition that circumscribes our topic, we refer to $(\mathcal{S}, \subset)$ as a *geometric containment order* when $\mathcal{S}$ is a nonempty and countable family of connected subsets of a finite-dimensional Euclidean space $\mathbb{R}^m$. The symbol $\subset$ denotes *proper* inclusion. For all specific cases of interest in the survey, the members of $\mathcal{S}$ have similar shapes (circles, spheres, convex n-gons, ...) and are closed in the usual topology of $\mathbb{R}^m$. With the exception of angle orders, the objects in $\mathcal{S}$ for particular cases are convex and compact.

Interesting classes of geometric containment orders are often named by the shape of the objects in their orders. For example, $(\mathcal{S}, \subset)$ is a *circle order* if every member of $\mathcal{S}$ is a closed circular disk in $\mathbb{R}^2$, $(\mathcal{S}, \subset)$ is an *n-gon order* if every member of $\mathcal{S}$ is a convex polygon with n vertices in $\mathbb{R}^2$, and $(\mathcal{S}, \subset)$ is a *box order in m dimensions* if every object in $\mathcal{S}$ is a box $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m]$ $(a_i \leq b_i)$ in $\mathbb{R}^m$ with edges parallel to the axes. An exception to this naming convention is the notion of an interval order (Fishburn, 1985; Trotter, 1992), which represents P by ordered intervals rather than by containment.

We denote by $2^{\mathbf{n}}$ the set of all subsets of $\{1, 2, \ldots, n\}$ ordered by proper inclusion, and by S_n the subposet of $2^{\mathbf{n}}$ for which $\mathcal{S}$ is the set of all singletons and their complements. S_n is often referred to as the *standard poset* of order dimension n. Figure 1 pictures the Hasse diagram of S_4 at the top along with containment representations for angular regions, squares, and circular disks.

Questions addressed in the survey for a class $\mathcal{C}$ of geometric containment orders include:

1. Are there interesting characterizations of the orders in $\mathcal{C}$ in terms of properties of posets $P = (X, \prec)$ that do not refer directly to the geometry of its objects?
2. Are all posets $P = (X, \prec)$ of a specific type members of $\mathcal{C}$?
3. What minimal posets $P = (X, \prec)$ are not in $\mathcal{C}$?
4. Are all members of $\mathcal{C}$ contained in another class $\mathcal{C}'$ of geometric containment orders?
5. What are the order dimensions of members of $\mathcal{C}$?
6. What are the crossing numbers of members of $\mathcal{C}$?
7. Is $\mathcal{C}$ closed under order composition, order duality, or the addition of a new minimum element to each order?

Definitions of terms used in the questions appear in the next few paragraphs. We then comment briefly on the history of our topic and conclude the introduction with an outline of ensuing sections. It should be remarked that a poset P is regarded as a geometric containment order of a specific type if it is representable as that type

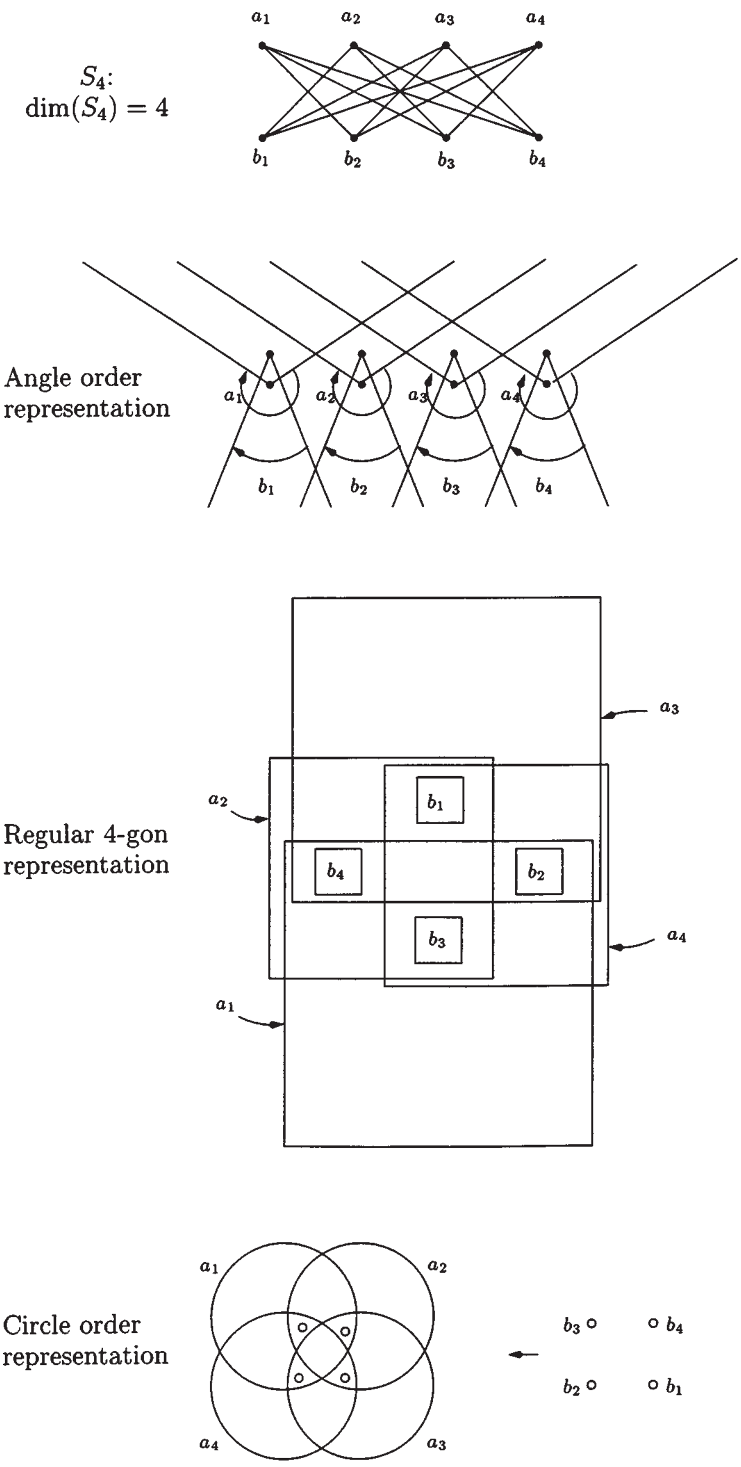

Figure 1. Standard poset of order dimension 4.

by (1.1). Although the mapping f used there need not be a bijection because order-equivalent members of X might be mapped into the same object in $\mathcal{S}$, bijective mappings can be presumed either by removing order-equivalent duplicates from X or by expanding $\mathcal{S}$ by adding perturbed copies of objects that are inclusion-equivalent to original objects.

We use P or $(X, \prec)$ throughout to denote a *poset* or *order* for which $\prec$ is an irreflexive and transitive binary relation on *ground set* X. It is assumed that $|X| \geq 3$ and that X is countable. The poset is *finite* if X is finite. The *dual* of $(X, \prec)$ is $(X, \prec^d)$, where $x \prec^d y$ if $y \prec x$, and the *zero-augmentation* of $(X, \prec)$ is $(X \cup \{0\}, \prec')$ where $0 \notin X$, $0 \prec' x$ for all $x \in X$, and the restriction of $\prec'$ to X equals $\prec$. A class $\mathcal{C}$ of posets is closed under duality, or *invertible*, if the dual of every order in $\mathcal{C}$ is in $\mathcal{C}$, and it is closed under the addition of 0, or *zero-augmentable*, if the zero-augmentation of every order in $\mathcal{C}$ is in $\mathcal{C}$. The *order composition* of posets $P = (X, \prec)$ and $Q = (Y, \prec')$ is the poset with ground set $X \times Y$ and order relation $\prec_0$ defined by $(x, y) \prec_0 (x^*, y^*)$ if $x \prec x^*$ and $y \prec' y^*$. Class $\mathcal{C}$ is *closed under order composition* if the order composition of P and Q is in $\mathcal{C}$ whenever $P, Q \in \mathcal{C}$. The *Cartesian product* $P \times Q$ of P and Q is defined like order composition except that $(x, y) \prec_0 (x^*, y^*)$ if $x \preceq x^*$ and $y \preceq' y^*$ with $=$ for at most one component. The *Cartesian product of orders* $P_1 = (X_1, \prec_1), \ldots, P_K = (X_K, \prec_K)$ is the poset $\times P_k$ with ground set $X_1 \times \cdots \times X_K$ and order relation $\prec_0$ defined by $(x_1, \ldots, x_K) \prec_0 (y_1, \ldots, y_K)$ if $x_k \preceq_k y_k$ for $k = 1, \ldots, K$ and $x_k \prec_k y_k$ for at least one k.

We say that $(X, \prec)$ is an *interval order* if $\{a \prec x, b \prec y\} \Rightarrow \{a \prec y \text{ or } b \prec x\}$ for all $a, b, x, y \in X$, which is true (Fishburn, 1970, 1985) if and only if there is a mapping I from X onto a set of closed and bounded real intervals such that

$$\forall x, y \in X, \quad x \prec y \Leftrightarrow \sup I(x) < \inf I(y). \tag{1.2}$$

Tanenbaum (1996) characterizes the *pairs* of finite posets $\{(X, \prec_1), (X, \prec_2)\}$ for which the same I mapping satisfies (1.1) for $(X, \prec_1)$ by interval inclusion and satisfies (1.2) for $(X, \prec_2)$ by interval precedence. Related results for so-called codominance pairs of posets are in Tanenbaum and Whitesides (1996).

Poset $(X, \prec)$ is a *linear order* (or chain) if $x \prec y$ or $y \prec x$ for all distinct x and y in X. A linear order $(X, \prec')$ is a *linear extension* of $(X, \prec)$ if $\prec \subseteq \prec'$. The *order dimension* $\dim(P)$ of $P = (X, \prec)$ is the minimum cardinality of a set of linear extensions of P the intersection of whose order relations equals $\prec$. Because every $(x, y) \in X \times X$ whose components are *incomparable* $\{x \neq y, \text{not}(x \prec y), \text{not}(y \prec x)\}$ has a linear extension in which $x \prec' y$ (Szpilrajn, 1930), $\dim(P)$ is well defined.

Order dimension is explored in depth in Trotter (1992). It is known that $\dim(S_n) = \dim(\mathbf{2^n}) = n$ (Dushnik and Miller, 1941; Komm, 1948), that $S_n = (X, \prec)$ for $n \geq 4$ is the only order with $|X| \leq 2n$ that has $\dim(X, \prec) \geq n$ (Bogart and Trotter, 1973), that $\dim(P)$ can be arbitrarily large for a finite interval order (Bogart, Rabinovitch and Trotter, 1976), and that all posets of dim $\leq 2m$ for

$m = 1, 2, \ldots$ are characterized by the following natural extension of Theorem 1 (Golumbic, 1984; Golumbic and Scheinerman, 1989).

THEOREM 2. $\dim(P) \leq 2m$ *if and only if* P *is* $\mathcal{S}$*-representable when* $\mathcal{S}$ *is the set of boxes in* $\mathbb{R}^m$ *with edges parallel to the axes.*

This is conveniently abbreviated by saying that $\dim(P) \leq 2m$ if and only if P is a box order in m dimensions. Similar abbreviations are used later.

The order dimension of a poset is a measure of its nonlinearity. Our final two definitions introduce other complexity measures used in studies of geometric containment orders. The first, from Golumbic, Rotem and Urrutia (1983), is the crossing number $\text{crs}(P)$ of a poset.

Let $P = (X, \prec)$ be a poset with n points $x_1, x_2, \ldots, x_n$ in its ground set. Let F_P be the set of all $(f_1, f_2, \ldots, f_n)$ in which each f_i is a continuous real-valued function on $[0, 1]$ with all $f_i(0)$ distinct and all $f_i(1)$ distinct such that, for all distinct i and j:

$\left|\{\lambda \in [0, 1] : f_i(\lambda) = f_j(\lambda)\}\right|$ is finite;

f_i and f_j cross if they touch;

$x_i \prec x_j \Leftrightarrow f_i(\lambda) < f_j(\lambda)$ for all $\lambda \in [0, 1]$.

The curves for f_i and f_j cross if and only if x_i and x_j are incomparable. The *crossing number* of P is the number of crossings for a worst-case pair with a best-case sequence in F_P:

$$\text{crs}(P) = \min_{(f_1,\ldots,f_n)\in F_P} \max_{\{\{i,j\}: i\neq j\}} |\{\lambda \in [0, 1] : f_i(\lambda) = f_j(\lambda)\}|.$$

Lin (1994) gives a general treatment of the crossing number. Known properties include: $\text{crs}(P) = 1 \Leftrightarrow \dim(P) = 2$ (Sidney, Sidney and Urrutia, 1988); $\text{crs}(S_n) = 2$ for $n \geq 3$ (Golumbic, Rotem and Urrutia, 1983); $\text{crs}(P) \leq \dim(P) - 1$ (Golumbic, Rotem and Urrutia, 1983); for every $n \geq 1$ there is a P for which $\dim(P) = n$ and $\text{crs}(P) = n - 1$ (Sidney, Sidney and Urrutia, 1988); and, in fact, $\text{crs}(2^{\mathbf{n}}) = n - 1$ (Brightwell and Winkler, 1989).

The other complexity measure, due to Alon and Scheinerman (1988), is the degrees of freedom $\text{dof}(\mathcal{F})$ of a family $\mathcal{F}$ of sets. We say that $\mathcal{F}$ has k *degrees of freedom* if k is the smallest positive integer for which there is an injection $g \colon \mathcal{F} \to \mathbb{R}^k$, $g(A) = (g_1(A), \ldots, g_k(A))$, and a finite list $p_1, p_2, \ldots, p_t$ of polynomials in $2k$ real variables such that, for all $A, B \in \mathcal{F}$, $A \subset B$ can be determined by the signs of the $p_i(g_1(A), \ldots, g_k(A), g_1(B), \ldots, g_k(B))$ for $i = 1, \ldots, t$. For example, the set of all closed and bounded real intervals has dof ≤ 2 because, with $g([a, b]) = (a, b)$, the signs of $p_1(a, b, c, d) = a - c$ and $p_2(a, b, c, d) = b - d$ completely determine whether $[a, b] \subset [c, d]$. The following theorem in Alon and Scheinerman (1988) suggests the power of their notion to identify posets that are not certain types of containment orders.

THEOREM 3. *If* $\text{dof}(\mathcal{F}) \leq k$ *then there is a finite* P *with* $\dim(P) = k + 1$ *such that* P *is not* $\mathcal{F}$*-representable.*

Although Dushnik and Miller (1941) foreshadowed our subject by (1.1) and Theorem 1, geometric containment orders did not become an active area for research until the early 1980s. The earliest publications of that era were Golumbic, Rotem and Urrutia (1983), Fishburn and Trotter (1985), Santoro and Urrutia (1987), and Santoro, Sidney, Sidney and Urrutia (1987). These were followed by half a dozen articles in 1988, a similar number in 1989, and the first and thus far only survey (Urrutia, 1989). Many of the people involved at that time exchanged ideas during a two-week NATO Advanced Study Institute on Graphs and Order, organized by Ivan Rival and held in Banff, Canada in May of 1984. A principal legacy of the Banff conference was the question

Is every finite P with $\dim(P) = 3$ a circle order?

This question, which first appeared in print in Santoro and Urrutia (1987) and was recently settled in the negative (Felsner, Fishburn and Trotter, 1999), was a prime motivator for research on containment orders in the decade following Banff. Despite its resolution, very little is known about the smallest poset of order dimension 3 that is not a circle order.

The next section reviews what is known about angle orders, the only intensively studied class of geometric containment orders whose objects are not compact and not necessarily convex. Section 3 presents results for four classes of n-gon orders, Section 4 surveys a progression of results for circle orders, and Section 5 focuses on classes of sphere orders in $\mathbb{R}^m$ for $m \geq 3$. Section 6 discusses containment orders for vertices, edges and faces of graphs, planar graphs, and convex polytopes. Section 7 summarizes results associated with the notions of comparability graph invariants and dynamic isometric inclusion. We conclude in Section 8 with some open problems.

2. Angle Orders

An *angular region* $A \subseteq \mathbb{R}^2$ is a closed region bounded by a pair (r_1, r_2) of distinct rays from a vertex $v \in \mathbb{R}^2$ that contains all points swept out by rays from v in the clockwise direction from r_1 to r_2. Vertex v of A is unique unless the angle from r_1 to r_2 is π, in which case A is a closed half plane. We say that A is *little* if its angle from r_1 to r_2 is less than π, and *big* if its angle exceeds π. Hence A is convex if and only if it is little or a half plane.

A containment order $(\mathcal{S}, \subset)$ is an *angle order* if $\mathcal{S}$ is a set of angular regions in $\mathbb{R}^2$. Because the proofs of theorems in this section presumed finiteness, we assume that all posets referred to below are finite. The following theorem summarizes key results in Fishburn and Trotter (1985, 1990) and Fishburn (1989a).

THEOREM 4. $\dim(P) \leq 4 \Rightarrow P$ *is an angle order. All standard posets* (S_n) *and interval orders are angle orders, and some circle orders are not angle orders. The class of all angle orders is invertible but not zero-augmentable.*

Fishburn and Trotter (1985) also gave examples of angle orders that must use a big angular region, and angle orders that must use a little angular region. A construction based on 2^7 was noted to yield a 198-point P with $\dim(P) = 7$ that is not an angle order. Santoro and Urrutia (1987) subsequently proved that angle orders with only little angular regions have $\text{crs}(P) \leq 4$, and similarly for angle orders with only big angular regions, then used this to give a 64-point P with $\dim(P) = 6$ that is not an angle order. Trotter (1987) used a similar procedure with two disjoint copies of 2^5 to obtain a 64-point P with $\dim(P) = 5$ that is not an angle order, and Alon and Scheinerman (1988) observed that Theorem 3 for degrees of freedom also implies that some posets of order dimension 5 are not angle orders.

THEOREM 5. *Some P with* $\dim(P) = 5$ *are not angle orders.*

Because the standard posets are angle orders, there are angle orders of arbitrarily large order dimension.

3. Convex Polygon Orders

Let $\mathcal{P}_n$ denote the set of all convex polygons in the plane with n vertices and n sides, $n \geq 3$. We refer to members of $\mathcal{P}_n$ as *n-gons* and consider four types of n-gon orders for each n. Given $\mathcal{S} \subseteq \mathcal{P}_n$, $(\mathcal{S}, \subset)$ is:

1. A *regular n-gon order* if every member of $\mathcal{S}$ is a regular n-gon with a side between two lowest vertices parallel to the abscissa;
2. A *weak regular n-gon order* if every member of $\mathcal{S}$ is a regular n-gon;
3. A *$\boldsymbol{\theta}$ n-gon order* if $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ with $0 < \theta_i < \pi$ for each i and $\sum \theta_i = (n-2)\pi$, and every member of $\mathcal{S}$ has a lowest side parallel to the abscissa with interior corner angles, beginning at the right vertex of the lowest side and proceeding counterclockwise, of $\theta_1, \theta_2, \ldots, \theta_n$ radians;
4. An *n-gon order* if every member of $\mathcal{S}$ is an n-gon.

For $n = 3$, objects for type 1 are equilateral triangles (including interiors) with horizontal bases, objects of type 2 are equilateral triangles oriented arbitrarily, objects for type 3 are triangles with horizontal lowest sides and equal interior-angle sequences, and those for type 4 are all triangles. Given $\boldsymbol{\theta}$, and going counterclockwise from the horizontal base side 1, the k^{th} sides of all n-gons for type 3 are mutually parallel, $k = 1, 2, \ldots, n$. For containment orders, it turns out that the only thing that matters for type 3 is the parallel-sides feature, not the particular $\boldsymbol{\theta}$.

As in the preceding section, the results of the present section were proved under finiteness, so we assume that all posets referred to below are finite. We begin with $\mathcal{R}_n$, the class of all (finite) regular n-gon orders for $n \geq 3$. The following composite theorem summarizes regular n-gon results in Santoro and Urrutia (1987), plus a few observations in Urrutia (1989).

THEOREM 6. *$P \in \mathcal{R}_3 \Leftrightarrow \dim(P) \leq 3$, and $\dim(P) \leq 3 \Rightarrow P \in \mathcal{R}_n$ for all $n \geq 4$. For every $n \geq 3$:*

(i) *$\mathcal{R}_n$ is invertible, zero-augmentable, and closed under order composition;*
(ii) *$P \in \mathcal{R}_n \Rightarrow \mathrm{crs}(P) \leq 2$;*
(iii) *$P \in \mathcal{R}_n \Rightarrow \dim(P) \leq n$;*
(iv) *$S_n \in \mathcal{R}_n$.*

In addition, no $\mathcal{R}_n$ contains the 14-point 4-dimensional poset $2^{\mathbf{4}} \setminus \{\emptyset, \{1, 2, 3, 4\}\}$.

Thus, whereas all order dimension 3 posets are in $\mathcal{R}_3$, not all order dimension 4 posets are in $\mathcal{R}_4$, whose objects are squares with sides parallel to the axes. The following theorem (Fishburn, 1989b) gives a different picture when we admit rectangles with sides parallel to the axes, which obtains for $\boldsymbol{\theta}$ 4-gon orders when $\boldsymbol{\theta} = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$.

THEOREM 7. *For every $n \geq 3$ and all $\boldsymbol{\theta}$ that adhere to our earlier definition, $\dim(P) \leq n \Leftrightarrow P$ is a $\boldsymbol{\theta}$ n-gon order, and the class of all $\boldsymbol{\theta}$ n-gon orders is invertible and zero-augmentable.*

The other main result in Fishburn (1989b) says that as soon as we allow arbitrary orientations, even when all n-gons are regular, we lose invertibility and zero-augmentability.

THEOREM 8. *For every $n \geq 3$, neither the class of all weak regular n-gon orders nor the class of all n-gon orders is invertible or zero-augmentable.*

For our final n-gon theorem, we denote by $\mathcal{G}_n$ the class of all n-gon orders, $n \geq 3$.

THEOREM 9. *For every $n \geq 3$:*

(i) *$P \in \mathcal{G}_n \Rightarrow \mathrm{crs}(P) \leq 2$;*
(ii) *$\dim(P) \leq 2n \Rightarrow P \in \mathcal{G}_n$;*
(iii) *$P \notin \mathcal{G}_n$ for some P with $\dim(P) = 2n + 1$.*

Parts (i) and (ii) are proved in Sidney, Sidney and Urrutia (1988), where it was also noted that $P \notin \mathcal{G}_n$ for some P with $\dim(P) = 2n + 2$. The sharper (iii) is proved by an application of Theorem 3 in Alon and Scheinerman (1988) in view of the fact that n-gon orders have $2n$ degrees of freedom.

4. Circle Orders

Let $\mathcal{C}_2$ denote the set of all finite circle orders and $\mathcal{C}_2^+$ the set of all circle orders with countable ground sets. When $(X_1, <), \ldots, (X_K, <)$ are linearly ordered sets of real numbers ordered naturally, $X_1 \times \cdots \times X_K$ denotes their Cartesian product order: if $X_k = X$ for all k, we write the product order as X^K.

We begin with a list of results for $\mathcal{C}_2$: see also Theorem 4 and Sections 6 and 7. One new definition is needed. $(\mathcal{S}, \subset)$ is an *up-parabola order* (Scheinerman, 1992) if $A \in \mathcal{S} \Rightarrow A = \{(x, y) : y \geq ax^2 + bx + c\}$ for some $a, b, c \in \mathbb{R}$. All posets in the following theorem are assumed to be finite.

THEOREM 10. *$\mathcal{C}_2$ equals the set of up-parabola orders and contains every interval order. It is invertible, zero-augmentable, closed under order composition, and contains a poset which has no circle-containment representation in which every minimal element is assigned a circle of radius zero. In addition:*

(i) $P \in \mathcal{C}_2 \Rightarrow \text{crs}(P) \leq 2$;
(ii) $\dim(P) \leq 2 \Rightarrow P \in \mathcal{C}_2$;
(iii) $\{1, 2, 3\}^3 \in \mathcal{C}_2$;
(iv) $P \notin \mathcal{C}_2$ *for some* P *with* $\dim(P) = 3$;
(v) $\mathbf{2}^4 \setminus \{\emptyset, \{1, 2, 3, 4\}\} \notin \mathcal{C}_2$;
(vi) $S_n \in \mathcal{C}_2$ *for all* n.

Many people have contributed here. The up-parabola equivalence from Scheinerman (1992) notes one of several equivalent representations of the usual circle inclusion that has

$$(x_1, y_1, r_1) \subset (x_2, y_2, r_2) \Leftrightarrow \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq r_2 - r_1 \quad \text{and} \quad r_1 \neq r_2$$

when (x, y, r) denotes the circular disk with center (x, y) and radius r. Interval order inclusion is proved in Fishburn (1988), invertibility in Urrutia (1989) and Scheinerman (1991), zero-augmentability in Sidney, Sidney and Urrutia (1988), closure under order composition in Urrutia (1989), and the inability to always shrink minimal-element circles to points in Scheinerman and Tanenbaum (1997). Sources for the others are:

(i) Sidney, Sidney and Urrutia (1988);
(ii) obvious from Theorem 1;
(iii) Fon-Der-Flaass (1993);
(iv) Felsner, Fishburn and Trotter (1999);
(v) Sidney, Sidney and Urrutia (1988) and Brightwell and Winkler (1989);
(vi) Brightwell and Winkler (1989).

Knight (1995) discusses a nonstandard-analysis approach to (iv). Attempts to prove (iv) led to many other results, including those for $\dim(P) = 3$ noted below. The proof of (iv) uses Ramsey theory and is uninformative about the smallest P with $\dim(P) = 3$ that is not a circle order. In view of (iii), we note that Brightwell and Scheinerman (1993) say that it is not known whether $\{1, 2, 3, 4\}^3$ is a circle order. This was resolved affirmatively by El-Zahar and Fateen (1998), but the question of whether $\{1, 2, 3, 4, 5\}^3$ is a circle order remains open.

Theorem 1 implies that every countable P with $\dim(P) = 2$ is in $\mathcal{C}_2^+$. While (iv) remained open, the question of whether every countable P with $\dim(P) = 3$ is in $\mathcal{C}_2^+$ was resolved negatively by Scheinerman and Wierman (1988). The tightest results were obtained by Lin (1991). Let $\mathbb{N}$ denote the set of positive integers.

THEOREM 11. *Let* $P = \{(1,1), (1,2), (1,3), (2,1), (2,3)\} \times \mathbb{N}$. *Then* $\dim(P) = 3$, $P \notin \mathcal{C}_2^+$, *and if* $\emptyset \subset A \subseteq \mathbb{R}^2$ *with* $|A| \leq 4$ *then* $A \times \mathbb{N} \in \mathcal{C}_2^+$.

Let $\mathbb{Z}$ denote the set of all integers. Historically, Scheinerman and Wierman (1988) proved that $\mathbb{Z}^3 \notin \mathcal{C}_2^+$. They noted also that $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{N}$ is not in $\mathcal{C}_2^+$ for large n. Hurlbert (1988) then gave a shorter proof of $\mathbb{N}^3 \notin \mathcal{C}_2^+$. This was followed by Lin's results in Theorem 11. Independently, Fon-Der-Flaass (1993) also proved that $\{1, 2\} \times \{1, 2\} \times \mathbb{N} \in \mathcal{C}_2^+$ and $\{1, 2\} \times \{1, 2, 3\} \times \mathbb{N} \notin \mathcal{C}_2^+$.

5. Sphere Orders

A poset $(\mathcal{S}, \subset)$ is an *n-sphere order* if $\mathcal{S}$ is a set of spheres in $\mathbb{R}^n$. Let $\mathcal{C}_n$ denote the set of all finite n-sphere orders and $\mathcal{C}_n^+$ the set of all countable n-sphere orders. This section summarizes results that extend to $n \geq 3$ some results in the preceding section.

Initial research on sphere orders was motivated by consideration of causality in space-time manifolds (Bombelli, Lee, Meyer and Sorkin, 1976). A natural affinity between causality and sphere orders is described in Brightwell and Winkler (1989), Scheinerman (1992) and Meyer (1993). We focus here on sphere orders in their own right.

The foundational paper on sphere orders, Brightwell and Winkler (1989), proved that for each n there is a poset in $\mathcal{C}_{n+1}$ that is not in $\mathcal{C}_n$. For each $n \geq 1$ let T_{n+2} be the poset $(X, \prec)$ with ground set $X = \{A \subseteq \{1, 2, \ldots, n+2\} : 1 \leq |A| \leq n+1\}$ and

$$A \prec B \quad \text{if} \quad A \subset B \quad \text{and either} \quad |A| = 1 \quad \text{or} \quad |B| = n+1 \quad \text{(or both).}$$

Thus $T_3 = S_3$ and $T_4 = 2^{\mathbf{4}} \setminus \{\emptyset, \{1, 2, 3, 4\}\}$. The following theorem combines the Brightwell and Winkler results with the non-shrinkability theorem for $n \geq 3$ of Scheinerman and Tanenbaum (1997) and the theorem of Felsner, Fishburn and Trotter (1999) which says that some finite 3-dimensional posets are not sphere orders.

THEOREM 12. *For each* $n \geq 1$, $\dim(T_{n+2}) = n+2$, $\mathrm{crs}(T_{n+2}) = n+1$, $T_{n+2} \notin \mathcal{C}_n$ *and* $T_{n+2} \in \mathcal{C}_{n+1}$. *For each* $n \geq 3$ *there is a* $P \in \mathcal{C}_n$ *which has no* $\mathcal{C}_n$ *representation in which every minimal element has radius zero. There is a finite* P *with* $\dim(P) = 3$ *that is in no* $\mathcal{C}_n$.

Meyer (1993) independently proved a result similar to an implication of the Brightwell–Winkler theorem. Let $T'_{n+2} = (X, \prec)$ with $X = \{A : A \subseteq \{1, 2, \ldots, n+2\}\}$

and $A \prec B$ if $|A| = 1$ and $A \subset B$. Then $T'_{n+2} \in \mathcal{C}_{n+1}$ and, as $n \to \infty$, $\min\{k : T'_n \in \mathcal{C}_k\} \to \infty$.

Brightwell and Winkler (1989) mention that $2^{\mathbf{5}} \in \mathcal{C}_4$ and conjecture that some finite poset is in no $\mathcal{C}_n$. Felsner, Fishburn and Trotter (1999) verifies this conjecture. Fon-Der-Flaass (1993) decided the corresponding question for countable sphere orders.

THEOREM 13. $\{1, 2\} \times \{1, 2, 3\} \times \mathbb{N}$ *is in no* $\mathcal{C}_n^+$.

Additional results for sphere orders are discussed in the next section.

6. Incidence Orders

A *simple graph* $G = (V, E)$ is a set V of vertices and a set E of edges, each of which is a pair $\{u, v\}$ of distinct vertices. We assume that $|V| \geq 3$. A *hypergraph* $H = (V, E)$ is a vertex set V and an edge set E of subsets of V. If $|e| = 2$ for every $e \in E$, then H is a simple graph. The *incidence order* P_H of hypergraph $H = (V, E)$ has ground set $V \cup E$ with

$$x \prec y \quad \text{if} \quad x \in V, y \in E \quad \text{and} \quad x \in y.$$

We denote P_H by P_G when H is a simple graph.

The following results are due to Scheinerman (1993) for P_G and Schrijver (1993) for P_H.

THEOREM 14. *There are finite simple graphs G with arbitrarily large* $\dim(P_G)$, *but* $P_G \in \mathcal{C}_3$ *for all such graphs. For every finite hypergraph* H,

$$k = \max\{|e| : e \in E\} \Rightarrow P_H \in \mathcal{C}_{2k-1}.$$

A graph G is *planar* if it can be drawn in the plane so that each vertex is a point, each edge is a continuous noncrossing, nontouching curve between its points, and no edges cross or touch between vertices. Scheinerman (1991) proved a nice strengthening of Theorem 14 for planar graphs. This is joined in the following theorem by Schnyder's (1989) remarkable and surprising discovery that the class of finite planar graphs equals the class of finite graphs whose incidence orders have order dimension at most 3. The last sentence of the theorem is from Scheinerman and Tanenbaum (1997).

THEOREM 15. *The following are mutually equivalent for every finite simple graph G:*

(i) *G is planar;*
(ii) $\dim(P_G) \leq 3$;
(iii) $P_G \in \mathcal{C}_2$.

There is a finite simple planar graph G whose P_G *has no circle order representation in which all circles for vertices have radius zero.*

We conclude this section by noting relationships between and facts about certain planar graphs and vertex-edge-face incidence orders of convex polytopes in $\mathbb{R}^3$ that are developed in Brightwell and Scheinerman (1993) and Brightwell and Trotter (1993). A few definitions are needed.

A *convex polytope* in $\mathbb{R}^3$ is the convex hull of a finite number of points in $\mathbb{R}^3$. We consider the set $\mathcal{M}$ of convex polytopes in $\mathbb{R}^3$ that do not lie in planes. A *face* of $M \in \mathcal{M}$ is the intersection of a plane and M that contains noncollinear points of M and does not intersect the interior of M. Each two faces that intersect in a line segment define that intersection as an *edge* of M, and each two edges that intersect in a point define that point as a *vertex* of M. We treat vertices as singleton subsets of $\mathbb{R}^3$. The *full incidence order* of $M \in \mathcal{M}$ with vertex set V, edge set E and face set F is the poset P_M with ground set $V \cup E \cup F$ and

$$x \prec y \quad \text{if} \quad x \subset y.$$

The following theorem, from Brightwell and Trotter (1993), was motivated by Schnyder's (1989) striking equivalence between (i) and (ii) of Theorem 15 but is substantially stronger than his equivalence.

THEOREM 16. $\dim(P_M) = 4$ *for all* $M \in \mathcal{M}$. *If any vertex or face is removed from* P_M*, the remainder has order dimension* 3.

To connect this to planar graphs, we define a *face* of a planar drawing of a finite simple planar graph G as the closure of a maximal open region in $\mathbb{R}^2$ after the points in the vertices and edges of the drawing have been removed. There is one outer, unbounded face; the others are compact subsets of the plane. We let F denote the set of faces for a particular drawing.

A graph is *connected* if there is an overlapping sequence of edges $\{u, v_1\}$, $\{v_1, v_2\}, \ldots, \{v_k, v\}$ between any two distinct vertices u and v. A graph is 3-*connected* if the removal of any two vertices of the graph and their incident edges leaves a connected graph.

Let $\mathcal{P}$ denote the set of all finite simple 3-connected planar graphs. It is easily seen that the (V, E, F) inclusion structure of a particular drawing of $G \in \mathcal{P}$ does not depend on the drawing, in particular on which face is chosen as the outer face, so we refer to (V, E, F) as the vertex-edge-face structure for G itself. A theorem of Steinitz (1934) says that a triple (V, E, F) is the vertex-edge-face structure for a $G \in \mathcal{P}$ if and only if it is inclusion isomorphic to a (V, E, F) structure for a convex polytope $M \in \mathcal{M}$.

The *full incidence order* for $G \in \mathcal{P}$ with structure (V, E, F) is the poset Q_G with ground set $V \cup E \cup F$ and $x \prec y$ if $x \subset y$. By Steinitz's theorem and Theorem 16, $\dim(Q_G) = 4$, and if a face in F is deleted then the remainder has order dimension 3. Brightwell and Trotter (1997) prove that dimensionality no greater than 4 continues to hold when 3-connectedness is not presumed. Given 3-connectedness, let Q_G^- denote the remainder when the outer face of a planar drawing of G is deleted from F. The incidence structure of Q_G^- depends on which face is deleted, but any such deletion produces a circle order as proved in Brightwell and Scheinerman (1993).

THEOREM 17. *$Q_G^- \in \mathcal{C}_2$ for all Q_G^- for all $G \in \mathcal{P}$.*

Brightwell and Scheinerman (1993) derive this from a beautiful generalization of a theorem of Koebe (1935) [see also Sachs (1994)] which says that a planar graph can be represented by nonoverlapping circles, one for each vertex, so that two vertices form an edge of the graph if and only if their circles are tangent.

7. Dynamic Inclusion and Invariants

We conclude our survey of results with two other topics covered by Urrutia (1989) that bear on geometric containment orders. The first, from Santoro, Sidney, Sidney and Urrutia (1987, 1989), considers containment under isometric movements that preserve objects' shapes. The second, from Urrutia (1988), concerns invariant properties of orders that arise from the same comparability graph.

Given $A, B \subseteq \mathbb{R}^m$, we say that A is *isometrically included* in B and write $A \subseteq_I B$ if some isometric copy of A, obtained by the operations of rotation, translation, and reflection, is included in B. The main question addressed in Santoro, Sidney, Sidney and Urrutia (1987, 1989), is whether $\subseteq_I$ can be characterized for a family $\mathcal{T}$ of objects in $\mathbb{R}^m$ by a finite number of real-valued functions $f_1, f_2, \ldots, f_n$ on $\mathcal{T}$ in the dominance-order sense that, for all $A, B \in \mathcal{T}$,

$$A \subseteq_I B \Leftrightarrow f_i(A) \leq f_i(B) \quad \text{for } i = 1, \ldots, n. \tag{7.1}$$

For example, the family of all spheres in $\mathbb{R}^m$ can be characterized by (7.1) with $n = 1$ and $f_1(A) = \text{Volume}(A)$, and the family of all regular convex k-gons for each $k \geq 3$ is characterized by $f_1(A) = \text{Area}(A)$. However, as soon as we consider slightly less regular shapes, (7.1) can fail regardless of the value of n.

THEOREM 18. *Suppose $\mathcal{T}$ is one of the following: all rectangles in $\mathbb{R}^2$; all isosceles triangles in $\mathbb{R}^2$; all convex k-gons in $\mathbb{R}^2$, $k \geq 4$; all right circular cylinders in $\mathbb{R}^3$. Then, for every n, there do not exist $f_1, \ldots, f_n$ that satisfy* (7.1) *for all $A, B \in \mathcal{T}$.*

The result for rectangles is proved in Santoro, Sidney, Sidney and Urrutia (1987) where it is also noted that a denumerable number of f_i characterize rectangles in the manner of (7.1). The other results in Theorem 18 are from Santoro, Sidney, Sidney and Urrutia (1989) which has other relevant information on the topic.

We now consider comparability graph invariants. A finite simple graph $G = (V, E)$ is a *comparability graph* (Gilmore and Hoffman, 1964; Fishburn, 1985) if there is a poset $(V, \prec)$ such that

$$\forall u, v \in V, \quad (u \prec v \text{ or } v \prec u) \Leftrightarrow \{u, v\} \in E. \tag{7.2}$$

Let $P(G)$ be the set of all posets $(V, \prec)$ that satisfy (7.2) for a given G. Roughly speaking, a comparability graph invariant is a poset parameter that has the same value for all orders in $P(G)$, for every comparability graph G. A precise statement of results follows.

THEOREM 19. *For every finite simple comparability graph G:*

(i) $\dim(P)$ *is the same for all* $P \in P(G)$*;*
(ii) $\mathrm{crs}(P)$ *is the same for all* $P \in P(G)$*;*
(iii) *either no orders or all orders in* $P(G)$ *are interval orders;*
(iv) *either no orders or all orders in* $P(G)$ *are circle orders;*
(v) *for each* $n \geq 3$*, either no orders or all orders in* $P(G)$ *are regular* n*-gon orders.*

Trotter (1992, p. 62) notes that several sets of authors have been credited for (i), but attributes primary credit to Gallai (1967). Results (ii), (iv) and (v) are from Urrutia (1988), and (iii) is noted in Möhring (1985, p. 64).

8. Open Problems

Although many questions for geometric containment orders have been answered, interesting questions remain open for the shapes highlighted in our survey as well as others that have not been intensively studied. Some specific questions raised by prior work are:

1. What are the smallest posets that are not angle orders?
2. What is the smallest n for which $\{1, 2, \ldots, n\}^3$ is not a circle order?
3. Is $\mathbf{2^4} \setminus \{\emptyset, \{1, 2, 3, 4\}\}$ the smallest poset that is not a circle order?
4. Is $\mathbf{2^6}$ a sphere order in any dimension?
5. What are the smallest posets that are not sphere orders?
6. Is $\mathbf{2^4} \setminus \{\emptyset, \{1, 2, 3, 4\}\}$ the smallest poset in no $\mathcal{R}_n$?
7. Which results noted only for finite posets also hold for countable posets?

We have seen that Theorem 3 for degrees of freedom often identifies the minimum-dimensional poset that is not a containment order of a particular type, but it does not do this for circle orders. Is there a condition whose addition to Theorem 3 will distinguish between $\min\ \dim = \mathrm{dof} + 1$ and $\min\ \dim < \mathrm{dof} + 1$ for a minimum-dimensional poset that is not a containment order of a particular type? Are there other complexity measures besides order dimension, crossing number, and degrees of freedom that reveal interesting facets of geometric containment orders?

An example of a simple shape that has not been intensively studied for containment orders is the ellipse. Urrutia (1989) noted that isometric inclusion of ellipses can be characterized by two functions but not one for (7.1). Varieties of ellipses for containment orders include those with principal axis parallel to the abscissa, those with either axis parallel to the abscissa, and ellipses in general position. Are there significant differences among the corresponding ellipse orders, and how do they relate to containment orders for other simple planar shapes?

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