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A combinatorial approach to correlation inequalities ☆

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Abstract

In this paper, we initiate a combinatorial approach to proving correlation inequalities for finite partially ordered sets. A new proof is provided for the strong form of the XYZ theorem, due to Fishburn. We also use our method to give a new proof of a related correlation result of Shepp involving two sets of relations. Our arguments are entirely combinatorial in the sense that they do not make use of the Ahlswede/Daykin theorem or any of its relatives.

Keywords: Partially ordered set; Correlation

1. Introduction

There are a number of results in the theory of partially ordered sets that have the same flavor; that of *correlation inequalities*. The basic theme is to treat the set $E(P)$ of all linear extensions of the poset P as a probability space, with all elements equally likely, and investigate circumstances under which events are positively correlated in this space. The most famous of these results is the XYZ Inequality, proved by Shepp [9] in 1982. This states that, if x, y and z are three elements of P, then the events $x>y$ and $x>z$ are non-negatively correlated in $E(P)$.

Shepp's proof of the XYZ Inequality uses the FKG Inequality, while the subsequent proof of a stronger result by Fishburn [4] uses the more general Ahlswede/Daykin

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Four Functions Theorem [1]. While the proof of the Four Functions Theorem is not especially hard, there has been continuing interest in providing a more elementary proof of the XYZ Inequality. We give such a proof in this paper, yielding Fishburn's stronger form. We do not claim that our proof is significantly simpler than Fishburn's—it is certainly *not* simpler than Shepp's proof—however, we hope that our method will provide fresh insights into correlation inequalities in general, and might lead to proofs of some outstanding conjectures in this area. We also illustrate our approach with proofs of two other results, all previous proofs of which use the Four Functions Theorem or a relative.

We begin with a brief review of the notation for linear orders and linear extensions which we will use in this paper. Readers who are more familiar with this background material are encouraged to proceed immediately to the next section.

Let n be a positive integer. We let $[n]=\{1,2,\dots,n\}$, and when $0\leqslant k\leqslant n$, we let $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. To display a function f from $[n]$ to a set X explicitly, we write $f=[x_1,x_2,\dots,x_n]$ to indicate that $f(i)=x_i$ for each $i=1,2,\dots,n$. When X is a finite set with $|X|=n$, we consider a linear order on X to be a function $L:[n]\underset{\text{onto}}{\overset{1-1}{\to}}X$. For each $x\in X$, the unique integer $i\in[n]$ for which $L(i)=x$ is called the *height* of x in L and is denoted by $h_L(x)$. We write $x<y$ in L when $h_L(x)<h_L(y)$.

Now let P be a finite partially ordered set (poset) on a ground set X of cardinality n. A linear order L on X is a *linear extension* of P if $h_L(x)<h_L(y)$ whenever $x<y$ in P. We let $E(P)$ denote the family of all linear extensions of P.

For a distinct pair $x,y\in P$, let $E_P[x>y]=\{L\in E(P):\ x>y \text{ in } L\}$. Then the *probability* that x is over y, denoted by $\text{prob}[x>y]$, is defined by

$$\text{prob}[x>y]=\frac{|E_P[x>y]|}{|E(P)|}. \tag{1}$$

We shall extend this notation to more complex events when necessary: for instance $\text{prob}[x>y,x>z]$ denotes the proportion of linear extensions of P in which x is over both y and z.

When the poset P is clear from the context and S is a subset of the ground set of P, we use the notation $E(S)$ for the set of all linear extensions of the subposet determined by S.

In the remainder of the paper, we will be constructing functions mapping a *domain* set to a *range* set. To assist the reader in keeping track of the various sets and subsets, we will reserve the letters D and δ, sometimes with subscripts, for objects associated with the domain set, and we will reserve the letters R and ρ for objects associated with the range set.

When S is an r-element set of integers, and we say that the elements of S are labelled $s_1,s_2,\dots,s_r$, we always mean that $s_1<s_2<\cdots<s_r$.

2. The XYZ theorems of Shepp and Fishburn

For x, y and z distinct elements of a poset P, to say that the events $x>y$ and $x>z$ are *non-negatively correlated* in the space $E(P)$ of linear extensions of P means to say that

$$\text{prob}[x>y]\,\text{prob}[x>z] \leqslant \text{prob}[x>y, x>z].$$

Using the standard notation for conditional probability, we see that this is equivalent to saying that

$$\text{prob}[x>y] \leqslant \text{prob}[x>y|x>z].$$

provided $\text{prob}[x>z]>0$.

The following theorem, known as the "XYZ theorem", was first proved by Shepp [9] using the FKG inequality and a clever definition of a distributive lattice to produce an analogous result for the space of *order-preserving maps*, and then deriving the result for the space of linear extensions by taking limits.

Theorem 2.1. *Let x, y and z be distinct points in a poset P. Then*

$$\text{prob}[x>y]\,\text{prob}[x>z] \leqslant \text{prob}[x>y, x>z].$$

In [4], Fishburn used repeated applications of the Ahlswede–Daykin Four Functions Theorem [1], and some complex definitions of auxiliary distributive lattices, to prove the following "strong" version of the XYZ theorem.

Theorem 2.2. *Let $\{x, y, z\}$ be a 3-element antichain in a poset P. Then*

$$\text{prob}[x>y]\,\text{prob}[x>y] < \text{prob}[x>y, x>z]. \tag{2}$$

Since the inequality is trivial if any pair among the three elements is related in P, Fishburn's strong version yields Shepp's XYZ theorem as a corollary. As we explain later, Fishburn not only showed that the inequality is strict except in trivial cases, but also quantified the minimum extent of the correlation.

We shall give a new proof of Theorem 2.2 in Section 4, using a method which avoids the use of the Four Functions Theorem. In the next section, we give a proof of a lemma used by Fishburn in [4]. Our new proof does not rely on this lemma, but it does provide a simple illustration of our general method.

In Section 5, we give a variant of our approach which yields a proof of another theorem of Shepp [8]. As pointed out by Brightwell [2], this result can be used to give a very quick proof of Fishburn's Lemma, and indeed our proof of Shepp's Theorem can be seen as a generalization of our proof of Fishburn's Lemma. One advantage of our new proof of Shepp's Theorem is that the cases of equality can be deduced quickly, which seems not to be the case for the original proof.

We finish by discussing some open problems for which our new method might prove useful.

3. A new proof of Fishburn's lemma

The starting point for this paper is a new proof of the following lemma of Fishburn [4], a result which played a key role in his proof of Theorem 2.2.

Lemma 3.1. *Let A and B be down-sets in a poset P. Then*

$$\frac{|E(A)|\,|E(B)|}{|E(A\cup B)|\,|E(A\cap B)|}\leqslant\frac{|A|!|B|!}{|A\cup B|!|A\cap B|!}. \tag{3}$$

Actually, we will find it convenient to prove the following result which is easily seen to be equivalent to Lemma 3.1.

Lemma 3.2. *Let A and B be down-sets in a poset P with $|A|=n$, $|B|=m$ and $|A\cap B|=k$. Then*

$$|E(A)|\,|E(B)|\binom{n+m}{n}\leqslant|E(A\cup B)|\,|E(A\cap B)|\binom{n+m}{n+m-k}. \tag{4}$$

Proof. Set

$$D=E(A)\times E(B)\times\binom{[n+m]}{n}$$

and

$$R=E(A\cup B)\times E(A\cap B)\times\binom{[n+m]}{n+m-k}.$$

Our aim is to prove that $|D|\leqslant|R|$.

For each triple $\delta=(L,M,S)\in D$, a function $\pi_\delta:[n+m]\to A\cup B$ can be derived naturally from δ by, loosely speaking, merging the linear orders L and M according to the template provided by S. More precisely, given $\delta=(L,M,S)$, with $S=\{s_1,s_2,\dots,s_n\}$ and $[n+m]-S=\{t_1,t_2,\dots,t_m\}$, define the function π_δ by setting:

- $\pi_\delta(i)=L(j)$ whenever $i\in S$ and $i=s_j$,
- $\pi_\delta(i)=M(j)$ whenever $i\in[n+m]-S$ and $i=t_j$.

We set $\Pi_D=\{\pi_\delta:\ \delta\in D\}$ and refer to the functions in Π_D as *domain patterns*.

It is obvious that any $\pi\in\Pi_D$ satisfies the following properties.

(a) For each $u\in(A-B)\cup(B-A)$, there is a unique integer $u_1\in[n+m]$ so that $\pi(u_1)=u$.
(b) For each $x\in A\cap B$, there are exactly two integers $x_1,x_2\in[n+m]$ with $x_1<x_2$ so that $\pi(x_1)=\pi(x_2)=x$.

In the remainder of the proof, whenever the pattern π is clear from the context, we will use the notation of (a) and (b) without comment.

We treat the set R in exactly the same way: for each triple $\rho=(J,K,W)\in R$, we define a function π_ρ mapping $[n+m]$ to $A\cup B$ by setting $W=\{w_1,w_2,\dots,w_{n+m-k}\}$ and

$[n+m]-W=\{z_1,z_2,\dots,z_k\}$, and defining π_ρ by

- $\pi_\rho(i)=J(j)$ whenever $i\in W$ and $i=w_j$,
- $\pi_\rho(i)=K(j)$ whenever $i\in[n+m]-W$ and $i=z_j$.

We let $\Pi_R=\{\pi_\rho\colon\ \rho\in R\}$ and refer to the functions in Π_R as *range patterns*. Range patterns also satisfy (a) and (b), and are subject to the same notational conventions.

From now on, we fix $\pi\in\Pi_D$, and consider the sets $\mathrm{Dom}(\pi)=\{\delta\in D\colon\ \pi_\delta=\pi\}$, and $\mathrm{Ran}(\pi)=\{\rho\in R\colon\ \pi_\rho=\pi\}$. By definition $\mathrm{Dom}(\pi)$ is non-empty. Our claim is that $|\mathrm{Dom}(\pi)|=|\mathrm{Ran}(\pi)|$: combining this result for every π evidently implies that $|D|\leqslant|R|$, as desired.

Given π, an element $\delta=(L,M,S)$ of $\mathrm{Dom}(\pi)$ is determined uniquely by the choice, for each element x of $A\cap B$, of whether S contains x_1 or x_2—this choice determines linear orders L and M on A and B, respectively, produced by restricting π to the sets S and $[n+m]-S$. We say that x is oriented *low–high* if $x_1\in S$, and *high–low* if $x_2\in S$. Of course, we are not normally free to choose all these orientations independently, as we also require that L and M are linear extensions. This produces constraints of the following types.

(i) For each $x\in A\cap B$ and $u\in A-B$, with $x<u$ in P and $x_1<u_1<x_2$, x must be low–high.
(ii) For each $x\in A\cap B$ and $u\in B-A$, with $x<u$ in P and $x_1<u_1<x_2$, x must be high–low.
(iii) For each $x,y\in A\cap B$, with $x<y$ in P and $x_1<y_1<x_2<y_2$, x and y must be oriented the same way.

Indeed, suppose for instance that we have $x<u$ in P for $x\in A\cap B$ and $u\in A-B$. If the order given by π is $u_1<x_1<x_2$, then either orientation of x will give $h_L(u)<h_L(x)$, so that L is not a linear extension of A—this contradicts the assumption that $\pi\in\Pi_D$. On the other hand if $x_1<x_2<u_1$, then either orientation fulfils the requirement that $h_L(x)<h_L(u)$. In the case where $x_1<u_1<x_2$, we will have $h_L(x)<h_L(u)$ if x_1 is in S, but not if x_2 is, i.e., if and only if x is oriented low–high: hence we get a constraint of type (i). The arguments in the other cases are similar. Note also that, since A and B are down-sets, the only possible relations in $A\cup B$ other than those covered by (i)–(iii) above are those inside $A-B$ or $B-A$, and these are necessarily respected by L and M regardless of the orientations of elements of $A\cap B$.

We say that $x\in A\cap B$ is *rigid* if there exists an element $u\in(A-B)\cup(B-A)$ for which $x<u$ and $x_1<u_1<x_2$. So for each rigid element x, the orientation of x is dictated by a constraint of type (i) or (ii).

Next, we define an auxiliary graph $\mathbf{G}=\mathbf{G}_\pi$ whose vertex set is $A\cap B$, with $\{x,y\}$ an edge in $\mathbf{G}$ whenever $x<y$ in P and $x_1<y_1<x_2<y_2$. The constraints of type (iii) can be summarized as saying that, for each component C of $\mathbf{G}$, all elements of C have the same orientation (we then say that C itself has this orientation). Since $\pi\in\Pi_D$, a component has at most one orientation: i.e., it does not contain both a rigid element that is forced to be low–high and one that is forced to be high–low.

Now let $A\cap B=C_1\cup C_2\cup\cdots\cup C_t$ be the partition of the vertex set of $\mathbf{G}$ into components. We say that a component is *rigid* if it contains a rigid element of $A\cap B$, and *free* otherwise.

Let r be the number of free components. It follows from our previous arguments that $\mathrm{Dom}(\pi)=2^r$, since an element of $\mathrm{Dom}(\pi)$ is specified uniquely by the orientations of the free components.

Let us turn now to describing $\mathrm{Ran}(\pi)$, for the same fixed $\pi\in\Pi_D$. Again, to specify an element $\rho\in\mathrm{Ran}(\pi)$ we must specify, for each element x of $A\cap B$, whether $x_1\in W$ (x is *low–high*) or $x_2\in W$ (*high–low*). The constraints on these choices are exactly the same as those in (i)–(iii) above, except that the type (ii) constraints now require the affected elements x to be low–high—thus all rigid elements are now forced to be low–high. The construction of $\mathbf{G}_\pi$ is unaltered by this change, as is the characterization of components as rigid (all elements are forced to be low–high) and free. Thus we have $|\mathrm{Ran}(\pi)|=2^r$.

This completes the proof. □

Let us make a couple of observations about the proof. First, note that we naturally obtain an explicit injection f from D to R, formed from a bijection $f_\pi:\mathrm{Dom}(\pi)\underset{\text{onto}}{\overset{1-1}{\to}}\mathrm{Ran}(\pi)$ for each $\pi\in\Pi_D$. Indeed, for $\delta=(L,M,S)\in\mathrm{Dom}(\pi)$, We determine a triple $(J,K,W)=f_\pi(\delta)\in R$ as follows.

(1) If $j\in[n+m]$ and $\pi(j)\in(A-B)\cup(B-A)$, then $j\in W$.
(2) If $x\in A\cap B$ and x belongs to a free component of the auxiliary graph $\mathbf{G}$, then $W\cap\{x_1,x_2\}=S\cap\{x_1,x_2\}$.
(3) If $x\in A\cap B$ and x belongs to a rigid component of the auxiliary graph $\mathbf{G}$, then $W\cap\{x_1,x_2\}=\{x_1\}$.
(4) From the preceding rules, we have determined a set $W\in\binom{[n+m]}{n+m-k}$. Let $W=\{w_1,w_2,\ldots,w_{n+m-k}\}$ and $[n+m]-W=\{z_1,z_2,\ldots,z_k\}$. Finally, define the linear orders J and K by setting $J(i)=\pi(w_i)$ for each $i\in[n+m-k]$ and $K(j)=\pi(z_j)$ for each $j\in[k]$.

Also, we should point out why our proof does not show that $|D|=|R|$. The reason is that there can be patterns in $\Pi_R-\Pi_D$, namely patterns π where there is a component of $\mathbf{G}_\pi$ containing both a rigid element x that is forced by a constraint of type (i) to be low–high in δ, and an element x' forced by a constraint of type (ii) to be high–low.

Here is the smallest such example. Consider the three element poset consisting of two maximal points a and b, each of which covers the element x. Then let $A=\{a,x\}$ and $B=\{b,x\}$. There are four patterns in Π_D, namely $\pi_1=[x,a,x,b]$, $\pi_2=[x,b,x,a]$, $\pi_3=[x,x,a,b]$ and $\pi_4=[x,x,b,a]$. Each of π_1 and π_2 has a trivial auxiliary graph with a single non-free component, so $|\mathrm{Dom}(\pi_1)|=|\mathrm{Dom}(\pi_2)|=|\mathrm{Ran}(\pi_1)|=|\mathrm{Ran}(\pi_2)|=1$. Furthermore, each of π_3 and π_4 have a trivial auxiliary graph with a single free component, so $|\mathrm{Dom}(\pi_3)|=|\mathrm{Dom}(\pi_4)|=|\mathrm{Ran}(\pi_3)|=|\mathrm{Ran}(\pi_4)|=2$. The two patterns $\pi_5=[x,a,b,x]$ and $\pi_6=[x,b,a,x]$ belong to $\Pi_R-\Pi_D$.

4. A new proof of the strong XYZ theorem

Let X denote the ground set of the poset P, so that $n=|X|\geqslant 3$. Following Fishburn, we observe that $E_P[x>y]=E_P[x>y,x>z]\cup E_P[z>x>y]$, $E_P[x>z]=E_P[x>y,x>z]\cup E_P[y>z>z]$, and $E(P)=E_P[z>x>y]\cup E_P[y>x>z]\cup E_P[x>y,x>z]\cup E_P[y>x,z>x]$. It follows that inequality (2) is equivalent to

$$|E_P[z>x>y]|\,|E_P[y>x>z]|<|E_P[x>y,x>z]|\,|E_P[x<y,x<z]|. \tag{5}$$

We will also follow Fishburn in proving a somewhat stronger statement than that of Theorem 2.2. Define the quantity λ_n by

$$\lambda_n=\begin{cases}(n-1)^2/(n+1)^2, & n \text{ odd},\\ (n-2)/(n+2), & n \text{ even}.\end{cases}$$

Note that regardless of the parity of n, we always have $\lambda_n<1$. It is also easy to check that λ_n is increasing in n.

The remainder of the proof will be devoted to proving the following inequality:

$$|E_P[z>x>y]|\,|E_P[y>x>z]|\leqslant\lambda_n|E_P[x>y,x>z]||E_P[x<y,x<z]|. \tag{6}$$

As was pointed out by Fishburn [4], the poset consisting of an $(n-2)$-element chain in which x appears as close to the middle as possible, together with two other points, y and z, each incomparable to all others, shows that his inequality (6) is tight.

Our argument for proving that inequality (6) holds is similar to the one given in the preceding section; however, the concept of a pattern must be modified somewhat, and there is a rather involved technical calculation, which is deferred to an Appendix A.

Let L be a linear order on X, and set $A_L(x)=\{u\in X:\ h_L(x)<h_L(u)\}$ and $B_L(x)=\{u\in X:\ h_L(u)<h_L(x)\}$. Here we use the letters A and B to suggest *above* and *below*, respectively.

Then let $D=E_P[z>x>y]\times E_P[y>x>z]$, $R=E_P[x>y,x>z]\times E_P[x<y,x<z]$, and

$$\mathscr{D}=\{(B_L(x)\cap B_M(x),A_L(x)\cap A_M(x)):\ (L,M)\in D\}.$$

For each $(B,A)\in\mathscr{D}$, set $D(B,A)=\{(L,M)\in D:\ B_L(x)\cap B_M(x)=B, A_L(x)\cap A_M(x)=A\}$. Then there is a natural partition

$$D=\bigcup\{D(B,A):\ (B,A)\in\mathscr{D}\}.$$

Dually, we define

$$\mathscr{R}=\{(B_J(x)\cap B_K(x),A_J(x)\cap A_K(x)):\ (J,K)\in R\}.$$

For each $(B,A)\in\mathscr{R}$, set $R(B,A)=\{(J,K)\in R:\ B_J(x)\cap B_K(x)=B, A_J(x)\cap A_K(x)=A\}$. Then there is a natural partition

$$R=\bigcup\{R(B,A):\ (B,A)\in\mathscr{R}\}.$$

To complete the proof, it suffices to show that $|D(B,A)|\leqslant\lambda_n|R(B,A)|$ for every $(B,A)\in\mathscr{D}$. So for the remainder of the argument, we fix a pair $(B,A)\in\mathscr{D}$, setting $b=|B|$ and $a=|A|$.

Now we discuss *patterns*. In this section, the concept will be slightly different from the one discussed previously. Informally, a *domain pattern* results from taking a pair $(L,M)\in D(B,A)$ and identifying the occurrence of x in the two orders. The elements below x in the orders are then merged, as are the elements above x. The end result is a function $\pi:[2n-1]\to X$.

More formally, we define the set D_0 to consist of all 4-tuples of the form (L,M,S_1,S_2), where

(1) $(L,M)\in D(B,A)$;
(2) for q_1 and q_2 defined by setting $h_L(x)=b+q_1+1$ and $h_M(x)=b+q_2+1$, S_1 is a subset of $[2b+q_1+q_2]$ of size $b+q_1$, and S_2 is a subset of $[2a+q_1+q_2]$ of size $a+q_2$.

From each 4-tuple $\delta=(L,M,S_1,S_2)\in D_0$, we derive a function $\pi=\pi_\delta$ from $[2n-1]$ to X as follows:

(1) $\pi(2b+q_1+q_2+1)=x$.
(2) Label the elements of S_1 as $s_1,s_2,\dots,s_{b+q_1}$ and label the elements of $[2b+q_1+q_2]-S_1$ as $t_1,t_2,\dots,t_{b+q_2}$. Then for each $i\in[2b+q_1+q_2]$, if $i=s_j\in S_1$, then $\pi(i)=L(j)$, and if $i=t_j\in[2b+q_1+q_2]-S_1$, then $\pi(i)=M(j)$.
(3) Label the elements of S_2 as $s'_1,s'_2,\dots,s'_{a+q_2}$ and label the elements of $[2a+q_1+q_2]-S_2$ as $t'_1,t'_2,\dots,t'_{a+q_1}$. Then for each $i\in[2a+q_1+q_2]$, if $i=s'_j\in S_2$, then $\pi(2b+q_1+q_2+1+i)=L(b+q_1+1+j)$, and if $i=t'_j\in[2a+q_1+q_2]-S_2$, then $\pi(2b+q_1+q_2+1+i)=M(b+q_2+1+j)$.

As before, we let $\Pi_D=\{\pi_\delta\colon\ \delta\in D_0\}$ denote the set of all *domain patterns*.

For $\pi=\pi_\delta\in\Pi_D$ and $u\in X-\{x\}$, there are two elements $u_1,u_2\in[2n-1]$ such that $\pi(u_1)=\pi(u_2)=u$ and $u_1<u_2$. Suppose $\pi=\pi_\delta$ where $\delta=(L,M,S_1,S_2)$, and define q_1 and q_2 as above. For each u, exactly one of u_1 and u_2 is in $S_1\cup S_2$: if it is u_1, then we say u is oriented *low–high* in δ; if it is u_2, we say u is oriented *high–low*. Each of the q_1 elements of $B_L(x)\cap A_M(x)$ is oriented low–high in δ, while each of the q_2 elements of $A_L(x)\cap B_M(x)$ is high–low. Note also that, while the sets A and B can be determined from the domain pattern, the numbers q_1 and q_2 prescribing the height of x in L and M cannot—although their sum $q_1+q_2=n-a-b-1$ can.

This discussion is repeated in an analogous manner to determine a set R_0 of 4-tuples of the form (J,K,W_1,W_2) with $(J,K)\in R(B,A)$. From each $\rho=(J,K,W_1,W_2)$, we derive a function $\pi=\pi_\rho$ mapping $[2n-1]$ to X. We then let $\Pi_R=\{\pi_\rho\colon\ \rho\in R_0\}$ denote the set of all *range patterns*. As before, it will turn out that $\Pi_D\subseteq\Pi_R$, but that in general there are range patterns that are not domain patterns.

As in the preceding section, given a domain pattern $\pi\in\Pi_D$, we set $\mathrm{Dom}(\pi)=\{\delta\in D_0\colon\ \pi_\delta=\pi\}$ and $\mathrm{Ran}(\pi)=\{\rho\in R_0\colon\ \pi_\rho=\pi\}$.

Now let $\pi\in\Pi_D$. We describe $\mathrm{Dom}(\pi)$ and $\mathrm{Ran}(\pi)$, showing in particular that the two sets have the same size. As in the previous section, we define an auxiliary graph $\mathbf{G}_\pi$, now with vertex set $X-\{x\}$. However, the rule for adjacency is much the same: if $u,v\in X-\{x\}$ and $u<v$ in P, then $\{u,v\}$ is an edge in $\mathbf{G}_\pi$ if and only if $u_1<v_1<u_2<v_2$.

As before, it is clear that, for any $\delta \in \mathrm{Dom}(\pi)$ and any component C of $\mathbf{G}_\pi$, all elements of C have the same orientation in δ—we deem the component to be oriented *low–high* or *high–low* in δ accordingly. Indeed, for fixed π, the only other constraints on the orientations of the elements are that y is necessarily low–high and z high–low in any $\delta \in \mathrm{Dom}(\pi)$.

This shows that, for $\pi \in \Pi_D$, y and z are in different components of $\mathbf{G}_\pi$. (Typically, there are range patterns π in which y and z are in the same component of $\mathbf{G}_\pi$—these are not domain patterns.) If $C_1, \dots, C_t$ are the components of $\mathbf{G}_\pi$, labelled so that y is in C_{t-1} and z in C_t, then an element $\delta \in \mathrm{Dom}(\pi)$ is specified uniquely by the orientations of the components $C_1, \dots, C_{t-2}$. Hence $|\mathrm{Dom}(\pi)| = 2^{t-2}$. Similarly, to choose an element $\rho \in \mathrm{Ran}(\pi)$, we must orient both C_{t-1} and C_t low–high, and we may orient the other components freely. Hence also $|\mathrm{Ran}(\pi)| = 2^{t-2}$.

So far, the argument has been very similar to that in the previous section. However, the quantities we have to deal with are not simply $|D_0| = \sum_{\pi \in \Pi_D} |\mathrm{Dom}(\pi)|$ and $|R_0| = \sum_{\pi \in \Pi_R} |\mathrm{Ran}(\pi)|$, since different pairs $(L, M) \in D(B, A)$ give rise to different numbers of 4-tuples (L, M, S_1, S_2) in D_0, and similarly for R_0. Indeed, for $(L, M) \in D(B, A)$, with q_1 and q_2 defined as before, there are $\binom{2b+q_1+q_2}{b+q_1}\binom{2a+q_1+q_2}{a+q_2}$ ways to choose S_1 and S_2, each of which gives an element of D_0.

To count $D(B, A)$, we thus need to attach a *weight* of $\binom{2b+q_1+q_2}{b+q_1}^{-1}\binom{2a+q_1+q_2}{a+q_2}^{-1}$ to each element $\delta = (L, M, S_1, S_2)$ of D_0. We therefore have

$$|D(B,A)| = \sum_{\pi \in \Pi_D} \sum_{\delta \in \mathrm{Dom}(\pi)} \frac{(b+q_1)!(b+q_2)!}{(2b+q_1+q_2)!} \frac{(a+q_1)!(a+q_2)!}{(2a+q_1+q_2)!}.$$

Recall that $q_1 = q_1(\delta)$ and $q_2 = q_2(\delta)$ depend on δ, although $q_1 + q_2 = n - a - b - 1$ depends only on (B, A). Similarly

$$|R(B,A)| = \sum_{\pi \in \Pi_R} \sum_{\rho \in \mathrm{Ran}(\pi)} \frac{(b+q_1)!(b+q_2)!}{(2b+q_1+q_2)!} \frac{(a+q_1)!(a+q_2)!}{(2a+q_1+q_2)!}.$$

Here $q_1 = q_1(\rho)$ and $q_2 = q_2(\rho)$ depend on ρ, being defined by $h_J(x) = b + q_1 + 1$ and $h_K(x) = b + q_2 + 1$, where $\rho = (J, K, W_1, W_2)$. Again $q_1 + q_2 = n - a - b - 1$.

It therefore suffices to show that, for each $\pi \in \Pi_D$

$$\sum_{\delta \in \mathrm{Dom}(\pi)} (b + q_1(\delta))!(b + q_2(\delta))!(a + q_1(\delta))!(a + q_2(\delta))!$$
$$\leqslant \lambda_n \sum_{\rho \in \mathrm{Ran}(\pi)} (b + q_1(\rho))!(b + q_2(\rho))!(a + q_1(\rho))!(a + q_2(\rho))!. \tag{7}$$

Although $|\mathrm{Dom}(\pi)| = |\mathrm{Ran}(\pi)|$, there seems to be no easy way to establish inequality (7) via a 1–1 correspondence between the terms of the two sums. Instead, we establish the inequality via a very natural 2–2 correspondence. (In the case where $t = 2$ and the sets each contain a single element, this will amount to double counting the single term in each sum.)

We have seen that the elements of $\delta \in \mathrm{Dom}(\pi)$ are in 1–1 correspondence with choices of orientation for the components $C_1, C_2, \dots, C_{t-2}$, and that the same statement holds for the elements $\rho \in \mathrm{Ran}(\pi)$. Given δ, let δ' denote the element of $\mathrm{Dom}(\pi)$ obtained from δ by orienting each of these $t-2$ components in the opposite manner. Then let ρ be the element of $\mathrm{Ran}(\pi)$ obtained by orienting each of $C_1, C_2, \dots, C_{t-2}$ as in δ, and let ρ' be the element obtained by orienting each component as in δ'. Of course, C_{t-1}, the component containing y, is oriented low–high in all four cases, while C_t is oriented high–low in δ and δ' and low–high in ρ and ρ'.

We claim that, for any $\delta = (L, M, S_1, S_2) \in \mathrm{Dom}(\pi)$,

$$\begin{aligned} &(b+q_1(\delta))!(b+q_2(\delta))!(a+q_1(\delta))!(a+q_2(\delta))! \\ &\quad + (b+q_1(\delta'))!(b+q_2(\delta'))!(a+q_1(\delta'))!(a+q_2(\delta'))! \\ &\qquad \leqslant \lambda_n((b+q_1(\rho))!(b+q_2(\rho))!(a+q_1(\rho))!(a+q_2(\rho))! \\ &\qquad\quad + (b+q_1(\rho'))!(b+q_2(\rho'))!(a+q_1(\rho'))!(a+q_2(\rho'))!), \end{aligned} \tag{8}$$

which will clearly imply inequality (7) and hence the full result.

To establish the claim, we next need to see how the various q_i are related. Let q_y be the number of elements in the component C_{t-1} that are in $B_L(x) \cap A_M(x)$. Let $r = q_1(\delta) - q_y$, so r counts the elements of $B_L(x) \cap A_M(x)$ that are in one of the components $C_1, \dots, C_{t-2}$ (any such component is oriented low–high in δ). Similarly let $q_z = |C_t \cap A_L(x) \cap B_M(x)|$ and $s = q_2(\delta) - q_z$. Observe that $q_1(\delta') = q_y + s$, $q_2(\delta') = q_z + r$, $q_1(\rho) = q_y + q_z + r$, $q_2(\rho) = s$, $q_1(\rho') = q_y + q_z + s$, $q_2(\rho') = r$.

Therefore the result will follow from the technical lemma below, whose straightforward proof is deferred to Appendix A.

Lemma 4.1. *For non-negative integers n, a, b, $q_y \geqslant 1$, $q_z \geqslant 1$, r and s with $a + b + q_y + q_z + r + s = n - 1$, we have*

$$\begin{aligned} &(b+q_y+r)!(b+q_z+s)!(a+q_y+r)!(a+q_z+s)! \\ &\quad + (b+q_y+s)!(b+q_z+r)!(a+q_y+s)!(a+q_z+r)! \\ &\qquad \leqslant \lambda_n((b+q_y+q_z+r)!(b+s)!(a+q_y+q_z+r)!(a+s)! \\ &\qquad\quad + (b+q_y+q_z+s)!(b+r)!(a+q_y+q_z+s)!(a+r)!). \end{aligned}$$

5. Shepp's theorem for disjoint unions

Let P be a poset with ground set X. Given a set $Q \subset X \times X$ with x incomparable to y in P for all $(x, y) \in Q$, we extend our previous notation by setting

$$E_P[Q] = \{L \in E(P):\ x < y \text{ in } P \text{ for all } (x, y) \in Q\}.$$

In this section, we provide a combinatorial proof of the following theorem of Shepp [8]. Shepp's approach was again to use the FKG inequality to prove the analogous result for order-preserving maps, and to derive this result in the limit.

Theorem 5.1. *Suppose that the ground set X of a poset P is the disjoint union of Y and Z, and that y and z are incomparable in P whenever $y \in Y$ and $z \in Z$. For arbitrary subsets Q_1, Q_2 of $Y \times Z$, we have*

$$|E_P[Q_1]|\,|E_P[Q_2]| \leqslant |E_P[Q_1 \cup Q_2]|\,|E(P)|.$$

Proof. Let $|X| = n$. Set $D = E_P[Q_1] \times E_P[Q_2] \times \binom{[2n]}{n}$ and $R = E_P[Q_1 \cup Q_2] \times E(P) \times \binom{[2n]}{n}$. Our aim is to show the existence of an injection from D to R.

As in the earlier proofs, we begin by assigning to each triple $\delta = (L, M, S) \in D$ the function $\pi = \pi_\delta : [2n] \to X$, obtained by merging the linear orders L and M on X according to the template S. Again, we adopt the convention that, for every $x \in X$, the two elements of $\pi_\delta^{-1}(x)$ will be denoted by x_1 and x_2 with $x_1 < x_2$.

As before, we let $\Pi_D = \{\pi_\delta\colon\ \delta \in D\}$ denote the set of *domain patterns*. Dually, each triple $\rho \in R$ gives rise to a function $\pi_\rho : [2n] \to X$, and $\Pi_R = \{\pi_\rho\colon\ \rho \in R\}$ denotes the set of all *range patterns*. Also as before, for $\pi \in \Pi_D$ we set $\mathrm{Dom}(\pi) = \{\delta \in D\colon\ \pi_\delta = \pi\}$, and similarly for $\mathrm{Ran}(\pi)$. □

Claim 1. $\Pi_D \subseteq \Pi_R$ *and* $|\mathrm{Dom}(\pi)| \leqslant |\mathrm{Ran}(\pi)|$ *for every* $\pi \in \Pi_D$.

Proof. Let $\pi \in \Pi_D$. Choose a triple $\delta = (L, M, S) \in \mathrm{Dom}(\pi)$. We then use δ to define a coloring ϕ_δ of X using two colors, red and blue. An element y of Y is colored *red* by ϕ_δ if $y_1 \in S$, and *blue* otherwise, i.e., if $y_2 \in S$. Conversely, ϕ_δ colors $z \in Z$ red if $z_2 \in S$, and blue otherwise. (So red means that the more helpful choice, in terms of Q_1 and Q_2 being respected, appears in L_1.)

In what follows, we will develop a list of properties which ϕ_δ must satisfy whenever $\delta \in \mathrm{Dom}(\pi)$. To this end, we again define an auxiliary graph $\mathbf{G} = \mathbf{G}_\pi$. For $y < y' \in Y$, join y and y' by an edge in $\mathbf{G}$ if $y_1 < y'_1 < y_2 < y'_2$. Similarly, if $z < z' \in Z$, join z and z' by an edge in $\mathbf{G}$ if $z_1 < z'_1 < z_2 < z'_2$. There are no edges in $\mathbf{G}$ of the form $\{y, z\}$ with $y \in Y$ and $z \in Z$. Now, let $X = C_1, C_2, \ldots, C_t$ denote the partition of X into components, noting that each component is either a subset of Y or a subset of Z. For each $x \in X$, we let $C(x)$ denote the component containing x. Our coloring rules require that if $\delta \in \mathrm{Dom}(\pi)$, then ϕ_δ will assign all vertices in any component the same color, so we may speak of the color assigned by ϕ_δ to a component of $\mathbf{G}$.

We present a list of coloring requirements which must be satisfied by any coloring ϕ_δ when $\delta \in \mathrm{Dom}(\pi)$ in order to respect the relations $y < z$ given by the elements (y, z) of Q_1. For a pair $(y, z) \in Q_1$, the requirement depends on the order on the four integers y_1, y_2, z_1 and z_2. Since $\pi \in \Pi_D$, it is clear that we cannot have $z_1 < z_2 < y_1 < y_2$ and, as usual, the order $y_1 < y_2 < z_1 < z_2$ does not yield any restriction on the colors assigned to $C(y)$ and $C(z)$. Four cases remain, and in each case, it is straightforward to verify that the listed requirement(s) must be satisfied by ϕ_δ when $\delta \in \mathrm{Dom}(\pi)$.

Case 1. $z_1 < y_1 < z_2 < y_2$. $C(y)$ and $C(z)$ must both be colored red.

Case 2. $y_1 < z_1 < z_2 < y_2$. $C(y)$ must be colored red.

Case 3. $z_1 < y_1 < y_2 < z_2$. $C(z)$ must be colored red.

Case 4. $y_1 < z_1 < y_2 < z_2$. At least one of $C(y)$ and $C(z)$ must be colored red.

Dually, for each pair $(y,z)\in Q_2$, we have restrictions of exactly the same form, but this time with red replaced by blue.

It is easy to see that $|\mathrm{Dom}(\pi)|$ is then the number of colorings of the set of components of $\mathbf{G}_\pi$ satisfying the coloring requirements added by the pairs (y,z) from $Q_1\cup Q_2$. In turn, this can be interpreted as the number of solutions of a 2-SAT instance Φ constructed as follows. For each component C of $\mathbf{G}$, there is a variable v_C in Φ, which is to be thought of as True if C is colored red, and False if C is colored blue. The requirements for a coloring to correspond to an element of $\mathrm{Dom}(\pi)$ are all of the form of 2-SAT clauses. To be precise:

- A requirement that component C must be colored red is represented by a clause (v_C), while a requirement that C must be colored blue is represented by a clause $(\overline{v_C})$.
- A requirement that one of C and C' be colored red is represented by a clause $(v_C v_{C'})$, while a clause that one of C and C' be colored blue is represented by a clause $(\overline{v_C v_{C'}})$.

We employ an analogous strategy for colorings associated with triples from $\mathrm{Ran}(\pi)$. However, the 2-SAT instance Φ' now reflects a single color and is obtained from Φ by replacing all negative literals by the corresponding positive ones. Now $|\mathrm{Ran}(\pi)|$ is the number of solutions of the 2-SAT instance Φ'. $\square$

To complete the proof, we show that Φ' has at least as many solutions as Φ by establishing the following elementary claim.

Claim 2. *For any instance Θ of SAT, define Θ' by replacing all occurrences of the negative literal $\bar{u}$ by u. Then Θ' has at least as many solutions as Θ.*

Proof. Let α be a solution to Θ such that the assignment α' obtained from α by changing the truth value of u is also a solution to Θ. Then α and α' are also solutions to Θ'.

If α is any other solution to Θ, then a solution to Θ' can be obtained from α by resetting u to True if necessary.

Thus there is an injection from the set of solutions of Θ to the set of solutions of Θ'. With this observation, the proofs of Claim 2 and Theorem 5.1 are complete. $\square$

Brightwell [2] showed that the inequality in Theorem 5.1 is strict unless both Y and Z can be partitioned into incomparable pieces $Y=Y_1\cup Y_2$ and $Z=Z_1\cup Z_2$, so that $Q_i\subseteq Y_i\times Z_i$ for $i=1,2$. His proof is rather complicated, so it is interesting to note that this extension follows very quickly from our proof, as we now show.

Suppose that there are relations $(y,z)\in Q_1$ and $(y',z')\in Q_2$ with y and y' in the same component of the comparability graph of P. Now consider any linear extension $L\in E_P(Q_1\cup Q_2)$ in which all of Y precedes all of Z, any linear extension M of P where all of Z precedes all of Y, and $S=\{1,\ldots,n\}$. The range pattern π_ρ arising from $\rho=(L,M,S)$ can be described as $Y<Z<Z<Y$. We claim that this is not a domain

pattern. Indeed, whenever $x < x'$ in P, we have $x_1 < x'_1 < x_2 < x'_2$, so x and x' are adjacent in the auxiliary graph **G**. In particular, y and y' are in the same component C of **G**. If π_ρ is a domain pattern arising from some δ, then, in the associated coloring, C is red since $(y,z) \in Q_1$ and $y_1 < z_1 < z_2 < y_2$, while C is blue since $(y',z') \in Q_2$ and $y'_1 < z'_1 < z'_2 < y'_2$. This contradiction shows that π_δ is not a domain pattern, and so the inequality in Theorem 5.1 is strict.

6. Remaining challenges

It is disappointing to us that we have been unable to prove any new results using our combinatorial approach. However, we do feel that this is a realistic goal, and in this section we offer a few problems that may yield to this method.

We start with a conjecture of Daykin and Daykin [3]. Given a poset P with ground set $X = \{x_1, \ldots, x_n\}$, and posets $Q_1, \ldots, Q_n$, the *lexicographic sum* $\sum_P Q_i$ is defined by taking disjoint copies of the Q_i, and adding all relations of the form $u < v$, where $u \in Q_i$, $v \in Q_j$, and $x_i < x_j$ in P.

Conjecture 6.1. Let P_0 be a poset whose ground set X_0 is the disjoint union of two chains $Y_0 = \{x_1, \ldots, x_m\}$ and $Z_0 = \{x_{m+1}, \ldots, x_n\}$. (There may be some relations between Y_0 and Z_0 in P_0.)

Let $S_1, \ldots, S_n$ be any finite posets, let Y be the union of the ground sets of $S_1, \ldots, S_m$ and Z the union of the ground sets of $S_{m+1}, \ldots, S_n$. Set $P = \sum_{P_0} S_i$. Let Q_1 and Q_2 be arbitrary subsets of $Y \times Z$. Then

$$|E_P[Q_1]|\,|E_P[Q_2]| \leqslant |E_P[Q_1 \cup Q_2]|\,|E(P)|.$$

Note that the case where Y_0 and Z_0 each have one element is exactly Shepp's Theorem from the previous section. Also, the case where each S_i has just one element (i.e., the result for P_0, Y_0 and Z_0) is a result of Graham et al. [5]. This is therefore a common generalization of the two theorems.

Intuition strongly suggests that Conjecture 6.1 is true, but it has so far resisted proof. One reason for this is that the known proofs of Shepp's Theorem and of the Graham–Yao–Yao Theorem follow very different lines, with Shepp's proof arguing via order-preserving maps and all proofs of the Graham–Yao–Yao Theorem (one clean proof is due to Kleitman and Shearer [7]) arguing directly with linear extensions. Our new proof of Shepp's Theorem works with the linear extensions, so it seems to reasonable to hope that it might form a basis for a proof of Conjecture 6.1.

We now turn to a correlation inequality of a completely different type. Let P be a finite poset with ground set X, and fix $x \in X$. Define the sequence $h_1, h_2, \ldots, h_n$, where $n = |X|$, by

$$h_i = |\{L \in E(P)\colon\ h_L(x) = i\}|.$$

This sequence is called the *height sequence* of x.

The following theorem was originally proved by Stanley [10] using the Alexandrov/ Fenchel inequalities for mixed volumes (in fact, a much stronger result is proved).

Theorem 6.2. *Let P be a poset with ground set X and set $n=|X|$. Then for each $x \in X$, the height sequence $h_1, h_2, \ldots, h_n$ of x is log-concave, i.e.,*

$$h_i h_{i+2} \leqslant h_{i+1}^2$$

for all $i = 1, 2, \ldots, n-2$.

Stanley's proof of this result is both compact and elegant, and is the only one known. It would be very interesting to have an alternative proof, especially because the algebraic machinery of Stanley's proof obscures the structural properties of the poset, so for example, nothing seems to be known about the following natural questions.

Question 6.3. *Under what circumstances is it true that the inequality $h_i h_{i+2} \leqslant h_{i+1}^2$ is tight?*

Question 6.4. *If the inequality $h_i h_{i+2} \leqslant h_{i+1}^2$ is strict, what is the minimum size of the error term?*

(Compare with the relationship between the XYZ Inequality and Fishburn's strong form.) To date, we have only been able to make marginal progress in providing a combinatorial proof of Theorem 6.2. Specifically, we can settle the special case when the height sequence contains exactly three non-zero terms. However, our approach does not seem likely to extend to a proof for the general case, and we consider this effort a major challenge.

Another motivation comes from the following specific problem posed to us by Kahn [6]. For an element x of an n-element poset P, with height sequence $h_1, \ldots, h_n$, the *average height* of x is defined by

$$h(x) = \sum_{i=1}^{n} i h_i.$$

Conjecture 6.5. Let P be a poset with a ground set X of size n, and let x and y be distinct elements of X. Also, let m be an integer with $2 \leqslant m \leqslant n$. If $|\{z \in X\colon\ z \leqslant x$ or $z \leqslant y\}| = m$, then

$$\max\{h(x), h(y)\} \geqslant m - 1.$$

Kahn noted that it follows easily from the log-concavity of the height sequence that the conjecture holds when $m = n$. However, when $n > m$, log-concavity alone seems to allow the maximum of the two heights to fall all the way down to $m \log 2$. Kahn also noted a natural generalization of the question to $k \geqslant 2$ points.

Appendix A. Proof of Lemma 4.1

We start with the case where $r = s = 0$.

Claim A.1. *Let $n \geqslant 3$ and let b and a be non-negative integers. Also, let q_y and q_z be positive integers with $b + a + q_y + q_z + 1 = n$. Then the maximum value of the expression*

$$\Phi = \frac{(b+q_y)!(b+q_z)!(a+q_y)!(a+q_z)!}{(b+q_y+q_z)!b!(a+q_y+q_z)!a!}$$

is λ_n.

Proof. Choose values of b, a, q_y and q_z which maximize the expression Φ.

Now suppose that $q_y > 1$. Then modify the parameters by decreasing q_y by one and increasing a by one. Then the new value $\hat{\Phi}$ of the function is given by

$$\hat{\Phi} = \frac{(b+q_y+q_z)(a+q_z+1)}{(b+q_y)(a+1)}\,\Phi > \Phi.$$

The contradiction shows that $q_y = 1$ and also, by symmetry, that $q_z = 1$.

Thus

$$\Phi = \frac{(b+1)(a+1)}{(b+2)(a+2)}.$$

Now suppose that $|b-a| \geqslant 2$. Then without loss of generality, we may assume $b \geqslant a+2$. In this case, we modify the parameters by subtracting one from b and increasing a by one. Then the new value $\hat{\Phi}$ of the function is given by

$$\hat{\Phi} = \frac{b(a+2)}{(b+1)(a+3)}$$

so

$$\frac{\hat{\Phi}}{\Phi} = \frac{1+1/(a+1)(a+3)}{1+1/b(b+2)} > 1.$$

The contradiction shows that $|b-a| \leqslant 1$, as claimed.

For n odd, it follows that $b = a = (n-3)/2$, and so

$$\Phi = \frac{((n-3)/2+1)^2}{((n-3)/2+2)^2} = \frac{(n-1)^2}{(n+1)^2} = \lambda_n.$$

For n even, we may assume by symmetry that $b = (n-4)/2$ and $a = (n-2)/2$, which yields

$$\Phi = \frac{((n-4)/2+1)((n-2)/2+1)}{((n-4)/2+2)((n-2)/2+2)} = \frac{n-2}{n+2} = \lambda_n.$$

This completes the proof of the claim. □

We now turn to the general case of Lemma 4.1. Recall that we have to prove that

$$\begin{aligned}&(b+q_y+r)!(b+q_z+s)!(a+q_y+r)!(a+q_z+s)!\\&\quad+(b+q_y+s)!(b+q_z+r)!(a+q_y+s)!(a+q_z+r)!\\&\qquad\leqslant\lambda_n((b+q_y+q_z+r)!(b+s)!(a+q_y+q_z+r)!(a+s)!\\&\qquad\quad+(b+q_y+q_z+s)!(b+r)!(a+q_y+q_z+s)!(a+r)!),\end{aligned}\tag{A.1}$$

where all the parameters are non-negative integers, $q_y \geqslant 1$, $q_z \geqslant 1$, and $n = b+a+q_y+q_z+r+s+1$.

If both r and s are non-zero, we modify the parameters by subtracting one from both r and s while adding one to both b and a. Notice that these changes leave both sides of (A.1) unchanged. Therefore we may assume without loss of generality that $s=0$.

We may then rewrite the desired inequality (A.1) as

$$\begin{aligned}&(b+q_y)!(b+q_z)!(a+q_y)!(a+q_z)!\\&\quad((b+q_y+1)\dots(b+q_y+r)(a+q_y+1)\dots(a+q_y+r)\\&\quad+(b+q_z+1)\dots(b+q_z+r)(a+q_z+1)\dots(a+q_z+r))\\&\qquad\leqslant\lambda_n(b+q_y+q_z)!b!(a+q_y+q_z)!a!((b+q_y+q_z+1)\dots\\&\qquad\quad(b+q_y+q_z+r)(a+q_y+q_z+1)\dots(a+q_y+q_z+r)\\&\qquad\quad+(b+1)\dots(b+r)(a+1)\dots(a+r)).\end{aligned}$$

From Claim 1, we know that

$$\frac{(b+q_y)!(b+q_z)!(a+q_y)!(a+q_z)!}{(b+q_y+q_z)!b!(a+q_y+q_z)!a!}\leqslant\lambda_{n-r}\leqslant\lambda_n,$$

so it suffices to check that

$$\begin{aligned}&(b+q_y+1)\dots(b+q_y+r)(a+q_y+1)\dots(a+q_y+r)\\&\quad+(b+q_z+1)\dots(b+q_z+r)(a+q_z+1)\dots(a+q_z+r)\\&\qquad\leqslant(b+q_y+q_z+1)\dots(b+q_y+q_z+r)\\&\qquad\quad(a+q_y+q_z+1)\dots(a+q_y+q_z+r)\\&\qquad\quad+(b+1)\dots(b+r)(a+1)\dots(a+r).\end{aligned}$$

We have equality here if $q_z=0$, and it is evident that the derivative of the left-hand side with respect to q_z is uniformly at most that of the right-hand side. Therefore, the inequality holds for all positive q_z.

This completes the proof of Lemma 4.1. □

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