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Adjacency posets of planar graphs

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ABSTRACT

In this paper, we show that the dimension of the adjacency poset of a planar graph is at most 8. From below, we show that there is a planar graph whose adjacency poset has dimension 5. We then show that the dimension of the adjacency poset of an outerplanar graph is at most 5. From below, we show that there is an outerplanar graph whose adjacency poset has dimension 4. We also show that the dimension of the adjacency poset of a planar bipartite graph is at most 4. This result is best possible. More generally, the dimension of the adjacency poset of a graph is bounded as a function of its genus and so is the dimension of the vertex–face poset of such a graph.

1. Introduction

With a finite graph **G**, we associate two partially ordered sets (posets), called the *incidence poset* and the *adjacency poset*. Both are height 2 posets.

The incidence poset (also called the *vertex–edge poset*) has the vertices of the graph as minimal elements and the edges of the graph as maximal elements. Furthermore, a vertex x is less than an edge e in the incidence poset of the graph when x is one of the two endpoints of e. Interest in incidence posets was initiated with the following remarkable theorem due to Schnyder [12].

Theorem 1.1. *A graph is planar if and only if the dimension of its incidence poset is at most 3.*

When a graph is drawn on a surface without crossings, then we may also consider the vertex–edge–face poset, a poset of height 3. The following theorem is due to Brightwell and Trotter [4]. Simpler proofs were given in [7,8,11].

Theorem 1.2. *If a planar* 3*-connected graph* **G** *is drawn without edge crossings in the plane, then the dimension of the vertex–edge–face poset is* 4*. Furthermore, if any vertex or any face is removed, the dimension is lowered to* 3*.*

In a subsequent paper [5], Brightwell and Trotter extended the preceding theorem with the following result for planar graphs with loops and multiple edges allowed.

Theorem 1.3. *If a planar multigraph is drawn without crossings in the plane, then the dimension of the vertex–edge–face poset is at most* 4*.*

Efforts have been made to extend these results to surfaces of higher genus, but the fact that the dimension of the incidence poset of a complete bipartite graph is at most 4 implies that there are graphs of arbitrary genus whose incidence posets have bounded dimension.

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1.1. Adjacency posets

Motivated by connections between chromatic number and poset dimension we proposed in [10] investigating the adjacency poset of a finite simple[1] graph **G**. This poset **P** has $V' \cup V''$ as its set of points where $V' = \{v' : v \in V\}$ and $V'' = \{v'' : v \in V\}$ are copies of the vertex set V of **G**. Elements of V' are minimal in **P** and elements of V'' are maximal in **P**. Furthermore, $x' < y''$ in **P** if and only if xy is an edge in **G**. In particular, note that x' is incomparable to x'' in **P**, for every $x \in V$.

In this paper, we study the dimension of the adjacency posets of planar, outerplanar and planar bipartite graphs. Our principal results will be the following three theorems:

Theorem 1.4. *If* **P** *is the adjacency poset of a planar graph* **G**, *then* $\dim(\mathbf{P}) \leq 8$. *Furthermore, there exists a planar graph whose adjacency poset has dimension at least* 5.

Theorem 1.5. *If* **P** *is the adjacency poset of an outerplanar graph* **G**, *then* $\dim(\mathbf{P}) \leq 5$. *Furthermore, there exists an outerplanar graph whose adjacency poset has dimension at least* 4.

Theorem 1.6. *If* **P** *is the adjacency poset of a planar bipartite graph* **G**, *then* $\dim(\mathbf{P}) \leq 4$. *Furthermore, there exists a planar bipartite graph whose adjacency poset has dimension* 4.

More generally, we will show that for every non-negative integer g, there is an integer d_g such that the dimension of the adjacency poset of a graph of genus g is at most d_g.

Our presentation will require a few well known tools from dimension theory, and we briefly summarize these results here. For additional background material, we refer the reader to the monograph [14] and the survey paper [15].

1.2. Background material on posets

For **P** a poset, we let $\mathrm{Inc}(\mathbf{P})$ denote the set of all incomparable pairs of **P**. If $(x, y) \in \mathrm{Inc}(\mathbf{P})$ and L is a linear extension of **P**, we say that (x, y) is *reversed* in L (also L *reverses* (x, y)) when $x > y$ in L. When $S \subseteq \mathrm{Inc}(\mathbf{P})$, we say that S is *reversible* when there exists a linear extension L of **P** reversing all pairs in S. Recall that a *strict alternating cycle of length* k in **P** is a subset $S = \{(x_i, y_i) : 1 \leq i \leq k\} \subseteq \mathrm{Inc}(\mathbf{P})$ with $x_i \leq y_j$ in **P** if and only if $j = i + 1$ (cyclically), for all $i, j = 1, 2, \ldots, k$.

The following elementary lemma is stated for emphasis.

Lemma 1.7. *Let* **P** *be a poset and let* $S \subseteq \mathrm{Inc}(\mathbf{P})$. *Then* S *is reversible if and only if* S *does not contain any strict alternating cycles.*

Also, recall that an incomparable pair $(x, y) \in \mathrm{Inc}(\mathbf{P})$ is called a *critical* pair when (1) $z < x$ in **P** implies $z < y$ in **P**, for all $z \in X$, and (2) $w > y$ in **P** implies $w > x$ in **P**, for all $w \in X$. We let $\mathrm{Crit}(\mathbf{P})$ denote the set of all critical pairs of **P**.

The definition implies that a pair (x', y') is a critical pair in the adjacency poset **P** of **G** exactly if all neighbors of y in **G** are also neighbors of x in **G**. The same condition characterizes critical pairs (x'', y'') in **P**. Therefore, all critical pairs of the adjacency poset **P** of a graph **G** are of the form (x', y'') if and only if for all pairs (u, v) of vertices there is a vertex w such that vw is an edge but uw is not an edge. Note that when uv is an edge of **G**, vertex u may be chosen as the "private neighbor" w of v. If $\mathbf{G}^+$ is the graph obtained by adding a new neighbor of degree 1 to every vertex of a graph **G**, then it is ensured that the adjacency poset $\mathbf{P}^+$ of $\mathbf{G}^+$ only has critical pairs that are of the form (x', y'').

Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ be a family of linear extensions of **P**. We say $\mathcal{R}$ is a *realizer* of **P** when $P = L_1 \cap L_2 \cap \cdots \cap L_t$, i.e., $x \leq y$ in **P** if and only if $x \leq y$ in L_i for all $i = 1, 2, \ldots, t$.

The following basic result is a standard tool from dimension theory.

Proposition 1.8. *If* $\mathbf{P} = (X, P)$ *is a poset and* $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ *is a family of linear extensions of* **P**, *then* $\mathcal{R}$ *is a realizer of* **P** *if and only if for every critical pair* $(x, y) \in \mathrm{Crit}(\mathbf{P})$, *there is some* i *for which* $x > y$ *in* L_i.

Recall that the *dimension* of a poset **P** is the least positive integer t for which **P** has a realizer of cardinality t. A poset **P** has dimension 1 if and only if it is a chain (total order). For posets that are not chains, Proposition 1.8 implies that we can reformulate the definition of dimension as follows.

Proposition 1.9. *The dimension of a poset* **P** *which is not a chain is the least positive integer* t *for which there exist subsets* $S_1, S_2, \ldots, S_t$ *such that*

(1) $\mathrm{Crit}(\mathbf{P}) = S_1 \cup S_2 \cup \cdots \cup S_t$, *and*
(2) *for each* $i = 1, 2, \ldots, t$, S_i *is reversible.*

One central motivation for our interest in adjacency posets comes from the following elementary observation.

[1] No loops or multiple edges are allowed.

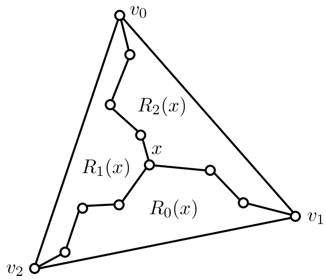

Fig. 1. Schnyder paths and regions.

Proposition 1.10. *Let* **P** *be the adjacency poset of a graph* **G***. The dimension of* **P** *is at least as large as the chromatic number of* **G***. Furthermore, whenever $W \subseteq V$, the set $S_W = \{(x', x'') : x \in W\}$ is reversible if and only if W is an independent set of vertices in* **G***.*

Proof. If x and y are distinct elements of V and are adjacent in **G**, then $\{(x', x''), (y', y'')\}$ is a strict alternating cycle of length 2, i.e., no linear extension can reverse both (x', x'') and (y', y''). This is enough to show that the dimension of **P** is at least as large as the chromatic number of **G**. On the other hand, if W is an independent set, then there are no strict alternating cycles contained in S_W, so S_W is reversible. □

In fact, when W is an independent set, we can say a bit more.

Proposition 1.11. *If* **P** *is the adjacency poset of a graph* **G** *and W is an independent set of vertices in* **G***, then the set $S = \{(x', y'') : x, y \in W\}$ is reversible.*

As we shall see, the dimension of the adjacency poset of a graph can in fact exceed the chromatic number of the graph. This can even happen for outerplanar graphs.

2. Background material on planar triangulations

Central to Schnyder's proof of Theorem 1.1 is a special coloring and orientation of the interior edges of a triangulation, today known as *Schnyder wood*. For existence and the theory of Schnyder woods we refer to [12,13,9,11]. Below we collect some of the features of Schnyder woods needed in our context.

Schnyder paths and regions

Let **T** be a planar triangulation in which the three exterior vertices are labeled v_0, v_1 and v_2 (in clockwise order). A Schnyder wood is an orientation and a coloring of the interior edges of **T**, using colors from $\{0, 1, 2\}$ such that:

(1) Each interior vertex has outdegree 3. Furthermore, these edges are colored (in clockwise order) 0, 1 and 2.
(2) If x is an interior vertex and $\alpha \in \{0, 1, 2\}$, there is a unique oriented path $P_\alpha(x)$ from x to v_α consisting of edges colored α.
(3) If x is an interior vertex and $\alpha \in \{0, 1, 2\}$, then x is the only vertex that $P_\alpha(x)$ and $P_{\alpha+1}(x)$ have in common.
(4) For each interior vertex, let $R_\alpha(x)$ be the region of the plane whose boundary consists of the edge $v_{\alpha+1}v_{\alpha+2}$ and the path $P_{\alpha+1}(x) \cup P_{\alpha+2}(x)$. If y is an interior vertex and $y \in R_\alpha(x)$, then $R_\alpha(y) \subseteq R_\alpha(x)$.
(5) If x and y are distinct interior vertices with $y \in R_\alpha(x)$, then $P_{\alpha+1}(y)$ intersects $P_{\alpha+2}(x)$ in at most one point and this occurs only when y is on $P_{\alpha+2}(x)$. Similarly, $P_{\alpha+2}(y)$ intersects $P_{\alpha+1}(x)$ in at most one point and this occurs only when y is on $P_{\alpha+1}(x)$.

We illustrate the concepts of Schnyder paths and regions in Fig. 1.

3. The upper bound for planar graphs

In this section, we prove that if **P** is the adjacency poset of a planar graph **G**, then $\dim(\mathbf{P}) \leq 8$. We will assume the following:

- **G** is a triangulation and all critical pairs in Crit(**P**) are of the form (x', y'') where $x' \in V'$ and $y'' \in V''$.

If this is not true for **G**, then we may add vertices with connecting edges to form a graph **H** satisfying the assumption such that **G** is an induced subgraph of **H**. As a consequence, the adjacency poset of **G** will be an induced subposet of the adjacency poset of **H**.

We consider a planar drawing of the maximal planar graph **G** and label the three exterior vertices in clockwise order as v_0, v_1 and v_2. We then consider a family of Schnyder paths and regions associated with this triangulation.

When x and y are distinct interior vertices and $y \in R_\alpha(x)$, we say that y is *properly contained* in $R_\alpha(x)$ when y does not lie on the path $P_{\alpha+1}(x) \cup P_{\alpha+2}(x)$. Note that when y is properly contained in $R_\alpha(x)$, any neighbor z of y is also contained in $R_\alpha(x)$.

The following elementary property is stated for emphasis. It is an immediate consequence of the properties of Schnyder paths and regions.

Remark 3.1. Let $\alpha \in \{0, 1, 2\}$, and let x, y and z be vertices of **G** with y properly contained in $R_\alpha(x)$ and z a neighbor of y in **G**. Then $R_\alpha(z) \subseteq R_\alpha(x)$; furthermore, if x and z are distinct, then $R_\alpha(z) \subsetneq R_\alpha(x)$.

3.1. Covering the set of critical pairs

Let $\phi : V \longrightarrow \{1, 2, 3, 4\}$ be a proper 4-coloring of **G**. For each $j = 1, 2, 3, 4$, we let $S_j = \{(x', x'') : \phi(x) = j\}$. As noted previously, each S_j is reversible.

We now define six subsets $S_5, S_6, \ldots, S_{10}$ of Crit(**P**) by the following rule: For each $\alpha = 0, 1, 2$,

(1) $S_{5+2\alpha} = \{(x', y'') \in \mathrm{Crit}(\mathbf{P}) : y \text{ is properly contained in } R_\alpha(x)\}$.
(2) $S_{5+2\alpha+1} = \{(y', x'') \in \mathrm{Crit}(\mathbf{P}) : y \text{ is properly contained in } R_\alpha(x)\}$.

Claim 3.2. *Each of $S_5, S_6, \ldots, S_{10}$ is reversible, i.e., none of these sets contains a strict alternating cycle.*

Proof. We first prove the claim for the sets $S_{5+2\alpha}$: Fix α and suppose that $S_{5+2\alpha}$ contains a strict alternating cycle $\{(x'_i, y''_i) : 1 \le i \le k\}$ for some $k \ge 2$. Since $x'_i \le y''_{i+1}$ in **P** (cyclically), we must have that $x_i y_{i+1}$ is an edge in **G**. But since y_{i+1} is properly contained in $R_\alpha(x_{i+1})$, this implies that x_i is contained in $R_\alpha(x_{i+1})$. Since x_i and x_{i+1} are distinct, we know that $R_\alpha(x_i) \subsetneq R_\alpha(x_{i+1})$, which cannot hold for all i. The contradiction completes the proof of these cases.

The proof for the sets $S_{5+2\alpha+1}$ is almost identical. We have $y'_i \le x''_{i+1}$ in **P**; hence, $y_i x_{i+1}$ is an edge in **G**. Since y_i is properly contained in $R_\alpha(x_i)$ we get $x_{i+1} \in R_\alpha(x_i)$. With $x_i \neq x_{i+1}$ this implies $R_\alpha(x_{i+1}) \subsetneq R_\alpha(x_i)$, which cannot hold for all i. □

It is easy to see that every critical pair in Crit(**P**) belongs to one of the sets in the family $\{S_1, S_2, \ldots, S_{10}\}$, and we have already noted that each of these ten sets is reversible. This shows that the dimension of **P** is at most 10.

3.2. Eliminating two of the ten

For each $i = 1, 2, \ldots, 10$, let L_i be a linear extension of **P** reversing all the critical pairs in the reversible set S_i.

For each $j = 1, 2, 3, 4$, let:

(1) $A_j = \{u' \in V' : \phi(u) \neq j\} \cup \{v'' \in V'' : \phi(v) = j\}$, and
(2) $B_j = \{u' \in V' : \phi(u) = j\} \cup \{v'' \in V'' : \phi(v) \neq j\}$.

Then $A_j \cup B_j = V' \cup V''$ for each $j = 1, 2, 3, 4$. Furthermore, for each $j = 1, 2, 3, 4$, there is no edge xy from **G** with $x' \in B_j$ and $y'' \in A_j$. It follows that we may form a linear extension $\hat{L}_j$ of **P** by the following construction:

(1) L_j^- is the induced ordering of L_9 on A_j,
(2) L_j^+ is the induced ordering of L_{10} on B_j,
(3) $\hat{L}_j$ is the concatenation of the two, i.e., $\hat{L}_j = L_j^- + L_j^+$.

Note that $\hat{L}_j$ reverses every critical pair from S_j for $j \in 1, 2, 3, 4$. A critical pair $(x', y'') \in S_9$ is reversed in $\hat{L}_{\phi(y)}$ and $(x', y'') \in S_{10}$ is reversed in $\hat{L}_{\phi(x)}$. This shows that the eight linear extensions in the family $\{\hat{L}_1, \hat{L}_2, \hat{L}_3, \hat{L}_4, L_5, L_6, L_7, L_8\}$ form a realizer of **P**. This completes the proof of the theorem.

4. The upper bound for outerplanar graphs

In this section we prove that if **P** is the adjacency poset of an outerplanar graph **G**, then $\dim(\mathbf{P}) \le 5$.

After adding vertices with connecting edges we have a 2-connected outerplanar graph **G**. Since 2-connected outerplanar graphs are Hamiltonian we may conclude that all critical pairs of the adjacency poset **P** of **G** are of the form (x', y'') where $x' \in V'$ and $y'' \in V''$.

By adding edges to **G** we obtain a maximal outerplanar graph **H**, i.e., **H** is an inner triangulation. Finally we add a root vertex r adjacent to all vertices in V. This yields a triangulation $\mathbf{H}^+$. Note that the original graph **G** will not in general be an induced subgraph of $\mathbf{H}^+$. Regardless, we have a maximal planar graph $\mathbf{H}^+$ for which we consider a planar drawing with $r = v_0$ as one of the three exterior vertices.

Since $\mathbf{H} = \mathbf{H}^+ \setminus \{r\}$ is outerplanar, it is 3-colorable. Let $\phi : V \longrightarrow \{1, 2, 3\}$ be a proper coloring of **H**. Then ϕ also determines a proper 3-coloring of **G**. So for each $i = 1, 2, 3$, the set $S_j = \{(x', x'') \in \mathrm{Crit}(\mathbf{P}) : \phi(x) = j\}$ is reversible.

Define subsets $S_5, S_6, \ldots, S_{10}$ of Crit(**P**) by the same definition as was used in the proof of the upper bound in Theorem 1.4. That is, we use a fixed Schnyder wood for $\mathbf{H}^+$ in the construction. Regardless of the choice of this Schnyder wood we note

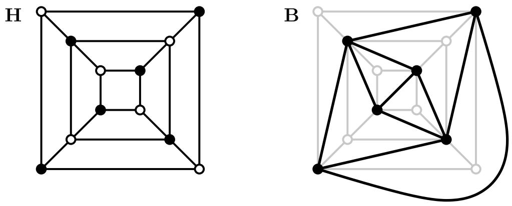

Fig. 2. A planar bipartite quadrangulation **H** and its associated graph **B**.

that having a vertex x in the proper interior of $R_0(y)$ for some $y \neq y$ would be in contradiction to the outerplanarity of **H**. Therefore S_5 and S_6 are both empty. It follows that $\mathrm{Crit}(\mathbf{P})$ is covered by the sets in the family $\{S_1, S_2, S_3, S_7, S_8, S_9, S_{10}\}$, and each of these seven sets is reversible. This shows that $\dim(\mathbf{P}) \leq 7$.

Let $L_1, L_2, L_3, L_7, L_8, L_9, L_{10}$ be linear extensions of **P** reversing all the critical pairs in their respective sets. As before, we can modify L_1, L_2 and L_3 to form new linear extensions $\hat{L}_1$, $\hat{L}_2$ and $\hat{L}_3$ so that $\{\hat{L}_1, \hat{L}_2, \hat{L}_3, L_7, L_8\}$ is a realizer of **P**. This shows that $\dim(\mathbf{P}) \leq 5$, and the proof is complete.

5. The upper bound for planar bipartite graphs

Our argument for this case requires two elementary results from dimension theory. First, when a poset **P** is the disjoint sum of two other posets, say $\mathbf{P} = \mathbf{Q} \cup \mathbf{R}$, then $\dim(\mathbf{P}) = \max(2, \dim(\mathbf{Q}), \dim(\mathbf{R}))$. Note that the preceding statement is just the special case of the formula for the dimension of a lexicographic sum when the base poset is a two-element antichain.

The second result is the trivial observation that a poset and its dual have the same dimension. With these remarks in mind, we can now proceed to prove that the dimension of the adjacency poset of a bipartite planar graph is at most 4.

Note that **G** is an induced subgraph of some 3-connected quadrangulation **H**. To see this we add vertices with connecting edges to **G** in three phases. In the first phase we make the graph 2-connected. In the second phase we insert stars in faces of higher order to get a quadrangulation. Finally we add four vertices to each face of the graphs so that the four new vertices together with the four vertices of the face induce a cube. The result is **H** and by construction the adjacency poset of **G** is an induced subposet of the adjacency poset of **H**.

Since **H** is bipartite, there is a partition $V = X \cup Y$ of the vertex set such that all edges in **H** have one endpoint in X and the other in Y. Then we note that the adjacency poset **P** is the disjoint sum of two height two posets $\mathbf{P}_1$ and $\mathbf{P}_2$. The elements of $\mathbf{P}_1$ are the minimal elements in $\{x' : x \in X\}$ together with the maximal elements in $\{y'' : y \in Y\}$. Similarly, the elements of $\mathbf{P}_2$ are the minimal elements in $\{y' : y \in Y\}$ together with the maximal elements in $\{x'' : x \in X\}$. Also, note that the posets $\mathbf{P}_1$ and $\mathbf{P}_2$ are dual. Since $\mathbf{P}_1$ is not a chain, it follows that $\dim(\mathbf{P}) = \dim(\mathbf{P}_1) = \dim(\mathbf{P}_2)$.

Next, we claim that $\mathbf{P}_1$ can be viewed as the vertex face poset of a planar graph **B**. The vertex set of **B** is X. Two vertices are joined by an edge if and only if they both belong to a quadrangular face of **H**; see Fig. 2. Since **H** is 3-connected, two faces of **H** can share at most one vertex from X. Therefore the graph **B** resulting from the construction is simple. It is also planar and its faces are in bijection to the elements of Y. Indeed there is an incidence between a vertex x and a face y in **B** exactly if x and y are adjacent in **H**, i.e., if $x' < y''$ in $\mathbf{P}_1$.

Since this new graph **B** is planar, it follows from Theorem 1.3 that the dimension of its vertex–face poset is at most 4. Since this poset is $\mathbf{P}_1$, the proof that the dimension of the adjacency poset of a planar bipartite graph is at most 4 is complete.

To see that this bound is best possible, consider the cube. The adjacency poset of the cube consists of two disjoint copies of the vertex–face poset of the tetrahedron. Among poset-theorists this poset is known as the *standard example* $\mathbf{S}_4$; see e.g. [14, page 12]. The dimension of $\mathbf{S}_4$ equals 4.

The upper bound of the adjacency poset of a planar bipartite graph immediately gives an upper bound to the dimension of height 2 posets with planar Hasse graph. We state this result as a corollary below.

Corollary 5.1. *If* **P** *is a height* 2 *poset and the underlying graph of the Hasse diagram of* **P** *is planar, then* $\dim(\mathbf{P}) \leq 4$*, and this bound is tight.*

Proof. Let $\mathbf{P}^d$ be the dual poset of **P**, and let **G** be the underlying graph of the Hasse diagram. Consider the poset $\mathbf{R} = \mathbf{P} \cup \mathbf{P}^d$. Note that we can regard **R** as the adjacency poset of the graph **G**, which is planar bipartite. By Theorem 1.6, we have $\dim(\mathbf{R}) \leq 4$, and the result follows from our earlier observations concerning disjoint sums and duals. We note that the upper bound is tight by considering **P** as the standard example $\mathbf{S}_4$ of a four-dimensional poset. This completes the proof. □

6. The lower bound for outerplanar graphs

In this section, we show that the dimension of the adjacency poset **P** of the outerplanar graph **G** shown in Fig. 3 has dimension at least 4. In fact, we show slightly more. In the spirit of the proofs of the upper bounds, we let S be the subset of $\mathrm{Crit}(\mathbf{P})$ consisting of all pairs of the form (x', y'') where $x' \in V'$ and $y'' \in V''$. We show that if $\mathcal{F}$ is a family of linear extensions of **P** and $\mathcal{F}$ reverses S, then $|\mathcal{F}| \geq 4$.

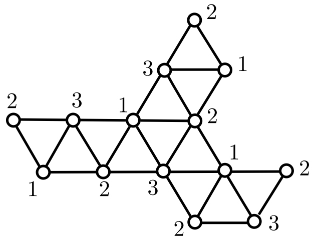

Fig. 3. An outerplanar graph.

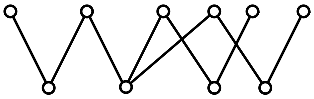

Fig. 4. A poset of height 2.

Actually, showing that reversing all critical pairs of the form (x', y'') requires four linear extensions implies a little more than just $\dim(\mathbf{P}) \geq 4$. Let $\mathbf{P}$ be a poset of height 2 such that the set S consisting of all critical pairs of the form (u, v) where u is a minimal element and v is a maximal element is not empty. The minimum number of linear extensions reversing S is the *interval dimension* of $\mathbf{P}$. This parameter was defined for general posets (arbitrary height) by Bogart and Trotter [3]. There are posets of large dimension and small interval dimension; this remains true for height 2.

We go for a contradiction. Suppose that there is a family $\mathcal{F} = \{L_1, L_2, L_3\}$ reversing all critical pairs in S.

Let $\phi : V \longrightarrow \{1, 2, 3\}$ be the proper 3-coloring of $\mathbf{G}$ defined by setting $\phi(x) = i$ when the critical pair (x', x'') is reversed in L_i. Without loss of generality, we may assume that ϕ is the 3-coloring shown in Fig. 3.

Claim 6.1. *For each $i = 1, 2, 3$, if x and y are distinct vertices with $\phi(x) = \phi(y) = i$, then the critical pairs (x', y'') and (y', x'') are both reversed in L_i.*

Proof. If $\mathbf{T}$ is the dual graph of $\mathbf{G}$ formed by the triangular faces (not including the exterior face), then $\mathbf{T}$ is a tree on 12 vertices. For distinct non-adjacent vertices u and v in $\mathbf{G}$, let $\rho(u, v)$ be the minimum distance in $\mathbf{T}$ between two faces, one containing u and the other containing v. Note that $\rho(u, v) \leq 8$ for all non-adjacent pairs u and v in $\mathbf{G}$.

We now prove the claim by induction on $\rho(x, y)$. Suppose first that $\rho(x, y) = 1$. Choose faces F_0 and F_1 that are adjacent in $\mathbf{T}$ such that $x \in F_0$ and $y \in F_1$. Then $|F_0 \cap F_1| = 2$. Note that for each $j \in \{1, 2, 3\}$ with $j \neq i$, there is a vertex u in $F_0 \cap F_1$ with $\phi(u) = j$. Then L_j reverses (u', u'') so it cannot reverse either (x', y'') or (y', x''). It follows that both (x', y'') and (y', x'') are reversed in L_i.

Now suppose that for some $k \geq 1$, the claim holds provided $\rho(x, y) \leq k$. Then consider a non-adjacent pair x, y with $\rho(x, y) = k + 1$. If $\phi(x) = \phi(y) = i$, then by inspection, we note that for each $j \in \{1, 2, 3\}$ with $j \neq i$, there exist vertices u and v with:

(1) u adjacent to x and v adjacent to y in $\mathbf{G}$.
(2) $\phi(u) = \phi(v) = j$.
(3) $\rho(u, v) \leq k - 1$.

It follows that L_j reverses all four of the critical pairs (u', u''), (u', v''), (v', v'') and (v', u''). However, this implies that neither (x', y'') nor (y', x'') is reversed in L_j. Hence, both are reversed in L_i. This proves the claim. $\square$

At this point in the argument, we consider the height 2 poset $\mathbf{Q}$ shown in Fig. 4.

Claim 6.2. *Let $\mathcal{F}$ be a family of linear extensions of $\mathbf{Q}$. If $\mathcal{F}$ reverses all critical pairs of $\mathbf{Q}$ of the form (u, v) where u is minimal and v is maximal in $\mathbf{Q}$, then $|\mathcal{F}| \geq 3$.*

We do not include the easy proof but remark that the claim is equivalent to the statement that $\mathbf{Q}$ has interval dimension 3. Moreover the removal of any point from $\mathbf{Q}$ lowers the interval dimension to 2. Felsner [6], has characterized all posets of height 2 with this property.

With the claim we can complete the proof. Note that the elements in $\{x' : \phi(x) = 3\} \cup \{y'' : \phi(y) = 2\}$ form a copy of the poset $\mathbf{Q}$ shown in Fig. 4. However, none of the critical pairs of the form $(x', y'') \in S$ with $\phi(x) = 3$ and $\phi(y) = 2$ is reversed in L_1. Hence, they must all be reversed by the family $\{L_2, L_3\}$, which is impossible.

7. The lower bound for planar graphs

Let $\mathbf{H}$ be the planar graph determined by attaching a new vertex r adjacent to all vertices in the outerplanar graph $\mathbf{G}$ shown in Fig. 3. We claim that the dimension of the adjacency poset of $\mathbf{H}$ is at least 5. To see this, let $\mathcal{R}$ be a realizer of the adjacency poset of $\mathbf{H}$. Choose a linear extension $L \in \mathcal{R}$ reversing the critical pair (r', r''), and let $\mathcal{F} = \mathcal{R} - \{L\}$.

Then $\mathcal{F}$ reverses the set S of critical pairs, and this requires $|\mathcal{F}| \geq 4$. Thus $|\mathcal{R}| = \dim(\mathbf{P}) \geq 5$.

8. Adjacency posets of graphs of higher genus

In this section, we show that the dimension of the adjacency poset of a graph is bounded as a function of the genus of the graph. More formally, we will prove the following result.

Theorem 8.1. *For every non-negative integer g, there exists an integer d_g such that if* $\mathbf{P}$ *is the adjacency poset of a graph* $\mathbf{G}$ *and the genus of* $\mathbf{G}$ *is g, then the dimension of* $\mathbf{P}$ *is at most d_g.*

Before beginning the proof, we assemble two necessary preliminary results. First, recall that the *acyclic chromatic number* of a graph $\mathbf{G}$ is the least positive integer t for which there is a proper coloring of $\mathbf{G}$ using t colors such that for every two colors, the subgraph of $\mathbf{G}$ induced by the vertices assigned these colors is acyclic.

The next theorem is due to Albertson and Bermen [1]. Alon et al. [2] have estimated the bound a_g as $O((2g-2)^{4/7})$.

Theorem 8.2. *For every non-negative integer g, there exists an integer a_g such that if* $\mathbf{G}$ *is a graph of genus g, then the acyclic chromatic number of* $\mathbf{G}$ *is at most a_g.*

Second, we have the following result due to Trotter and Moore [16].

Theorem 8.3. *Let* $\mathbf{P}$ *be a poset whose diagram is a tree (or a forest). Then the dimension of* $\mathbf{P}$ *is at most* 3.

Note that the diagram of the poset $\mathbf{Q}$ shown in Fig. 4 is a tree and it has dimension 3, so the preceding theorem is best possible.

With these results in mind, we are now ready to prove the theorem.

Proof of Theorem 8.1. The theorem holds when $g = 0$, so we may assume that g is positive. Let $\mathbf{G}$ be a graph of genus g, and let $\mathbf{P}$ be the adjacency poset of $\mathbf{G}$. We may add 'private neighbours' to some vertices of $\mathbf{G}$ to ensure that all critical pairs are of the form (x', y''). We show that $\dim(\mathbf{P}) \leq 3\binom{a_g}{2}$.

Let ϕ be a proper coloring of $\mathbf{G}$ using a_g colors such that for every two colors, the vertices assigned these two colors induce an acyclic subgraph of $\mathbf{G}$, i.e., a collection of trees.

For each pair $\{i, j\}$ of distinct colors $1 \leq i, j \leq a_g$, we consider the adjacency poset of the graph induced by vertices of colors i and j in $\mathbf{G}$. This subposet of $\mathbf{P}$ is a collection of disjoint trees. By Theorem 8.3, three linear extensions are enough to reverse the critical pairs in this subposet. Any linear extension of a subposet of a poset can be extended to a linear extension of the parent poset. So with $3\binom{a_g}{2}$ linear extensions, we can reverse all critical pairs in $\mathbf{P}$. □

The theorem has direct implications for the dimension of vertex–face posets of graphs of genus g.

Corollary 8.4. *The dimension of the vertex–face poset of a graph of genus g is at most d_g, where d_g is the bound from Theorem* 8.1.

Proof. The proof is similar to the argument for bipartite planar graphs, but the known bound is transferred in the other direction.

Let $\mathbf{G}$ be a graph embedded in a surface of genus g and let V, E and F denote the sets of vertices, edges and faces of $\mathbf{G}$. Construct $\mathbf{H}$ with vertex set $V \cup F$ and edges (v, f) for all incident pairs $v \in V$ and $f \in F$. This graph clearly comes with a drawing without crossings on the same surface as $\mathbf{G}$. Therefore, we know from Theorem 8.1 that the dimension of the adjacency poset $\mathbf{P}$ of $\mathbf{H}$ is at most d_g.

Since $\mathbf{H}$ is bipartite we know that $\mathbf{P}$ has two components. Let $\mathbf{P}_1$ be the component induced by V' and F''; clearly $\dim(\mathbf{P}_1) \leq d_g$. From the construction of $\mathbf{H}$ it follows that $\mathbf{P}_1$ is the vertex–face poset of $\mathbf{G}$. This completes the proof. □

9. Concluding remarks

We have some feeling that the lower bound is tight for outerplanar graphs, i.e., we make the following conjecture:

Conjecture 9.1. *If* $\mathbf{P}$ *is the adjacency poset of an outerplanar graph, then* $\dim(\mathbf{P}) \leq 4$.

We are less certain of the correct answer for planar graphs. Perhaps the right answer is 6.

For the genus, the right answer is likely to be $O(g)$.

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References

[1] M.O. Albertson, D.M. Bermen, An acyclic analogue to Heawood's theorem, Glasg. Math. J. 9 (1978) 163–166.
[2] N. Alon, B. Mohar, D.P. Sanders, On acyclic colorings of graphs on surfaces, Israel J. Math. 94 (1996) 273–283.

[3] K.P. Bogart, W.T. Trotter, On the complexity of posets, Discrete Math. 16 (1976) 71–82.
[4] G.R. Brightwell, W.T. Trotter, The order dimension of convex polytopes, SIAM J. Discrete Math. 6 (1993) 230–245.
[5] G.R. Brightwell, W.T. Trotter, The order dimension of planar maps, SIAM J. Discrete Math. 10 (1997) 515–528.
[6] S. Felsner, 3-Interval irreducible partially ordered sets, Order 11 (1994) 97–125.
[7] S. Felsner, Convex drawings of planar graphs and the order dimension of 3-polytopes, Order 18 (2001) 19–37.
[8] S. Felsner, Geodesic embeddings and planar graphs, Order 20 (2003) 135–150.
[9] S. Felsner, Geometric Graphs and Arrangements, Vieweg Verlag, 2004.
[10] S. Felsner, W.T. Trotter, Dimension, graph and hypergraph coloring, Order 17 (2000) 167–177.
[11] S. Felsner, F. Zickfeld, Schnyder woods and orthogonal surfaces, Discrete Comput. Geom. 40 (2008) 103–126.
[12] W. Schnyder, Planar graphs and poset dimension, Order 5 (1989) 323–343.
[13] W. Schnyder, Embedding planar graphs on the grid, in: Proceedings 1st ACM-SIAM Symp. Discr. Algo. 1990, pp. 138–148.
[14] W.T. Trotter, Combinatorics and Partially Ordered Sets: Dimension Theory, The Johns Hopkins University Press, Baltimore, 1992.
[15] W.T. Trotter, Partially ordered sets, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, Elsevier, Amsterdam, 1995, pp. 433–480.
[16] W.T. Trotter, J.I. Moore, The dimension of planar posets, J. Combin. Theory Ser. B 21 (1977) 51–67.