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A NOTE ON DILWORTH'S EMBEDDING THEOREM

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ABSTRACT. The dimension of a poset X is the smallest positive integer t for which there exists an embedding of X in the cartesian product of t chains. R. P. Dilworth proved that the dimension of a distributive lattice $L = \underline{2}^X$ is the width of X. In this paper we derive an analogous result for embedding distributive lattices in the cartesian product of chains of bounded length. We prove that for each $k \geq 2$, the smallest positive integer t for which the distributive lattice $L = \underline{2}^X$ can be embedded in the cartesian product of t chains each of length k equals the smallest positive integer t for which there exists a partition $X = C_1 \cup C_2 \cup \cdots \cup C_t$ where each C_i is a chain of at most $k - 1$ points.

1. Preliminaries. A poset consists of a pair (X, P) where X is a set and P is a reflexive, antisymmetric, and transitive relation on X. The notations $(x, y) \in P$ and $x \leq y$ in P are used interchangeably. If x and y are distinct points in X and neither (x, y) nor (y, x) is in P, then we say x and y are incomparable and write xIy. For convenience we will frequently use a single symbol to denote a poset. If X and Y are isomorphic posets, then we write $X = Y$ and if X is isomorphic to a subposet of Y, then we write $X \subseteq Y$. The dual of a poset X, denoted $\hat{X}$, is the poset on the same set with $x \leq y$ in $\hat{X}$ iff $y \leq x$ in X.

If (X, P) and (Y, Q) are posets, their free sum, denoted $X + Y$, is the poset $(X \dot{\cup} Y, P \dot{\cup} Q)$ where $\dot{\cup}$ denotes disjoint union. Their cartesian product $X \times Y$ is the poset $(X \times Y, S)$ where $S = \{((x, y), (z, w)): x \leq z \text{ in } X$ and $y \leq w$ in $Y\}$. The cartesian product of n copies of X is denoted X^n. The join of (X, P) and (Y, Q), denoted $X \oplus Y$, is the poset $(X \dot{\cup} Y, P \cup Q \cup X \times Y)$. A function $f: Y \to X$ is order preserving iff $y \leq w$ in Y implies $f(y) \leq f(w)$ in X. The cardinal power of X and Y, denoted X^Y, is the poset consisting of all ordering preserving functions from Y to X with $f \leq g$ in X^Y iff $f(y) \leq g(y)$ in X for every $y \in Y$.

A poset C for which $x, y \in C$ imply $x \leq y$ or $y \leq x$ is called a chain. We denote the n element chain $0 < 1 < 2 < \cdots < n - 1$ by $\underline{n}$. A chain (X, L) is said to be linear extension of (X, P) when $P \subseteq L$. We also say L is a linear extension of P. By a theorem of Szpilrajn [12], if $\mathcal{C}$ denotes the collection of all linear extensions of P, then $\bigcap \mathcal{C} = P$.

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A poset A for which $x, y \in A$ and $x \neq y$ imply xIy is called an antichain. We denote an element antichain by $\bar{n}$. The width of a poset X, denoted $W(X)$, is the number of elements in a maximum antichain in X.

The justification for the exponential notation for the cardinal power of posets is given by the following property (see [2] for details).

Fact 1. $X^{Y+Z} = X^Y \times X^Z$.

In this paper we are concerned primarily with cardinal powers of the form $\underline{2}^X$. For such posets, we have

Fact 2. $\underline{2}^{\underline{n}} = \underline{n+1}$ and $\underline{2}^{\bar{n}} = \underline{2}^n$.

If (X, P) and (Y, Q) are posets, $X = Y$, and $P \subseteq Q$, then it is easy to see that $\underline{2}^Y \subseteq \underline{2}^X$. In fact a stronger result holds.

Lemma 1. *Let* (X, P) *and* (Y, Q) *be posets,* $Y \subseteq X$, *and* $P \cap (Y \times Y) \subseteq Q$. *Then* $\underline{2}^Y \subseteq \underline{2}^X$.

Proof. Define a function $F: \underline{2}^Y \to \underline{2}^X$ by $F(f)(x) = f(x)$ if $x \in Y$, $F(f)(x) = 0$ if $x \in X - Y$ and there exists $y \in Y$ such that $y > x$ in X and $f(y) = 0$, and $F(f)(x) = 1$ otherwise. It is straightforward to verify that F is an embedding.

2. **Introduction.** Dushnik and Miller [5] defined the dimension of a poset X, denoted Dim X, as the smallest positive integer t for which there exist t linear extensions $L_1, L_2, \ldots, L_t$ of the partial ordering P on X such that $L_1 \cap L_2 \cap \cdots \cap L_t = P$. Ore [9] gave an equivalent definition of Dim X as the smallest positive integer t for which $X \subseteq C_1 \times C_2 \times \cdots \times C_t$ where each C_i is a chain.

A very important example of a poset is a distributive lattice for which we have the following well-known representation theorem: *A poset* M *is a distributive lattice iff* $M = \underline{2}^X$ *for some poset* X. In 1950, R. P. Dilworth [4] published the following theorem giving the dimension of a distributive lattice.

Theorem 1. Dim $\underline{2}^X = W(X)$.

In order to prove Theorem 1, Dilworth derived his famous decomposition theorem.

Theorem 2. *If* X *is a poset and* $W(X) = n$, *then the point set* X *can be partitioned into* n *subsets* $C_1, C_2, \ldots, C_n$ *such that the subposet determined by each* C_i *is a chain.*

Compact proofs of Theorem 2 appear in [10] and [15] and Theorem 1 is also discussed in [11].

In this paper we generalize the concept of dimension for posets to obtain an extension of Theorem 1. For an integer $k \geq 2$, we define the k-dimension of a poset X, denoted $\text{Dim}_k X$ as the smallest positive integer t for which $X \subseteq \underline{k}^t$.

3. **Some elementary inequalities.** In [13], the inequality $\operatorname{Dim}_2 X \le |X|$ for all X is established and the family of posets for which equality holds is determined. In [14], the inequalities $\operatorname{Dim}_3 X \le \{|X|/2\}$ for $|X| \ge 5$ and $\operatorname{Dim}_4 X \le [|X|/2]$ for $|X| \ge 6$ are established. Hiraguchi [6] proved that $\operatorname{Dim} X \le [|X|/2]$ for $|X| \ge 4$ and Bogart and Trotter [3] and Kimble [8] determined the collection of all posets for which equality holds.

Clearly $\operatorname{Dim} X \le \operatorname{Dim}_k X$ and since $\underline{k}^t \subseteq \underline{k+1}^t$, we have $\operatorname{Dim}_{k+1} X \le \operatorname{Dim}_k X$. Since there are k^t points in $\underline{k}^t$, we have $\operatorname{Dim}_k X \ge \log_k |X|$ and since the longest chain in $\underline{k}^t$ has length $(k-1)t+1$, we conclude $\operatorname{Dim}_k \underline{n} = \{(n-1)/(k-1)\}$. It is also easy to compute $\operatorname{Dim}_k \underline{n}$ by the methods compiled by Katona [6].

Theorem 3. $\operatorname{Dim}_k X \le 2 \operatorname{Dim}_{k+1} X$.

Proof. Suppose $\operatorname{Dim}_{k+1} X = t$ and let $f\colon X \to \underline{k+1}^t$ be an embedding. Define $g\colon X \to \underline{k}^{2t}$ by:

$$g(x)(i) = \begin{cases} f(x)(i) - 1 & \text{when } f(x)(i) > 0 \text{ and } i \le t, \\ 0 & \text{when } f(x)(i) = 0 \text{ and } i \le t, \\ f(x)(i) & \text{when } f(x)(i) < k \text{ and } i > t, \\ k-1 & \text{when } f(x)(i) = k \text{ and } i > t. \end{cases}$$

It follows easily that g is an embedding and thus $\operatorname{Dim}_k X \le 2t$.

In order to determine whether or not the inequality of Theorem 3 is best possible, we need the following generalization of a well-known property (see [2, problem 7, p. 101]) of dimension which we state without proof.

Fact 4. If X and Y are posets, then $\operatorname{Dim}_k X \times Y \le \operatorname{Dim}_k X + \operatorname{Dim}_k Y$. If X and Y have distinct greatest and least elements, then equality holds.

Since $\operatorname{Dim}_k \underline{k+1} = 2$ and $\operatorname{Dim}_{k+1} \underline{k+1} = 1$, it follows from Fact 4 that $\operatorname{Dim}_k \underline{k+1}^t = 2t$ while $\operatorname{Dim}_{k+1} \underline{k+1}^t = t$ for all $t \ge 1$.

4. **Dilworth's embedding theorem.** A short proof of Dilworth's embedding theorem (Theorem 1) is given here for the sake of completeness. We assume Theorem 2.

To show that $\operatorname{Dim} \underline{2}^X \le W(X)$, let $|X| = m$, $W(X) = n$, and $X = C_1 \cup C_2 \cup \cdots \cup C_n$ be a decomposition into chains. It follows that

$$\underline{2}^X \subseteq \underline{2}^{C_1 + C_2 + \cdots + C_n} = \underline{2}^{C_1} \times \underline{2}^{C_2} \times \cdots \times \underline{2}^{C_n} \subseteq \underline{m+1}^n$$

and thus $\operatorname{Dim} \underline{2}^X \le n$.

On the other hand if A is an antichain of X with $|A| = n$, then $\underline{2}^n = \underline{2}^A \subseteq \underline{2}^X$ and we conclude that $\operatorname{Dim} \underline{2}^X \ge \operatorname{Dim} \underline{2}^n = n$.

The reader is invited to compare this argument with the proof of Theorem 3 in [13].

5. Some additional inequalities. For a poset X and an integer $m \geq 1$, let $P_m(X)$ be the smallest positive integer t for which there exists a partition of the point set of X of the form $X = C_1 \cup C_2 \cup \cdots \cup C_t$ where the subposet determined by each C_i is a chain with $|C_i| \leq m$. The first half of the argument given in the preceding section allows us to conclude that $\operatorname{Dim}_k \underline{2}^X \leq P_{k-1}(X)$.

Now every poset Y can be written as the free sum $Y = Y_1 + Y_2 + \cdots + Y_r$ of its components. For a poset Y with components $Y_1, Y_2, \ldots, Y_r$ and an integer $m \geq 1$, we then define $S_m(Y) = \sum_{i=1}^r \{|Y_i|/m\}$. To provide a generalization of the concept of width, we define $W_m(X) = \max\{S_m(Y): Y \subseteq X\}$. Dilworth's decomposition theorem can then be restated in the following form.

Theorem 4. *For every poset X, there exists an integer m_0 such that $m \geq m_0$ implies $P_m(X) = W_m(X)$.*

To see the connection between these definitions and Dilworth's embedding theorem we observe that the following result holds.

Theorem 5. *For every poset X and every integer $k \geq 2$, $W_{k-1}(X) \leq \operatorname{Dim}_k \underline{2}^X \leq P_{k-1}(X)$.*

Proof. Choose a subposet $Y \subseteq X$ with $W_{k-1}(X) = S_{k-1}(Y)$; let the components of Y be $Y_1, Y_2, \ldots, Y_r$ and for each $i \leq r$ let C_i be a linear extension of Y_i. If follows that

$$\underline{2}^{C_1} \times \underline{2}^{C_2} \times \cdots \times \underline{2}^{C_r} \subseteq \underline{2}^{Y_1} \times \underline{2}^{Y_2} \times \cdots \times \underline{2}^{Y_r}$$

$$= \underline{2}^{Y_1 + Y_2 + \cdots + Y_r} = \underline{2}^Y \subseteq \underline{2}^X$$

and therefore

$$\operatorname{Dim}_k(\underline{2}^{C_1} \times 2^{C_2} \times \cdots \times \underline{2}^{C_r}) \leq \operatorname{Dim}_k \underline{2}^X.$$

$$\operatorname{Dim}_k(\underline{2}^{C_1} \times \underline{2}^{C_2} \times \cdots \times \underline{2}^{C_r}) = \sum_{i=1}^{r} \{|C_i|/(k-1)\} = \sum_{i=1}^{r} \{|Y_i|/(k-1)\}$$

$$= S_{k-1}(Y) = W_{k-1}(X).$$

For $m = 1$, $W_1(X) = P_1(X) = |X|$ for all X. It is also true that $W_2(X) = P_2(X)$ for all X; in fact a more general result holds which we outline here. For a graph H with components $H_1, H_2, \ldots, H_r$ let $S_m(H) = \sum_{i=1}^r \{|H_i|/m\}$. For a graph G, let $W_m(G) = \max\{S_m(H): H$ is an induced subgraph of $G\}$. Also let $P_m(G)$ be the smallest positive integer n for which there exists a partition of the vertex set of G into n subsets so that the induced subgraph spanned by each subset is a complete graph on at most m vertices.

For a poset X the comparability graph of X, denoted G_X, is the graph whose vertex set is the point set of X with distinct points $x, y \in X$ adjacent in G_X iff $x < y$ or $y < x$ in X. Clearly $P_m(X) = P_m(G_X)$ and $W_m(X) = W_m(G_X)$.

Theorem 6. $W_2(G) = P_2(G)$ *for all graphs.*

Proof. We assume Hall's matching theorem for graphs and then proceed by induction on $|X|$. Now suppose G is a graph with $W_2(G) = t$ and let H be a subgraph of G with components $H_1, H_2, \ldots, H_r$ so that $W_2(G) = W_2(H) = \Sigma_{i=1}^{r}\{|H_i|/2\} = t$. We further assume that H is chosen so that r is maximal and $|H|$ is minimal. Thus $W_2(H_i - x) < W_2(H_i)$ for every $i \leq r$ and every $x \in H_i$ and we may assume that $H \neq X$.

Now construct a bipartite graph (X, Y) with $X = \{v_1, v_2, \ldots, v_r\}$ and $Y = G - H$. A vertex $y \in Y$ is adjacent to v_i in (X, Y) iff y is adjacent to at least one vertex of H_i in G.

By Hall's matching theorem, there exists a matching of Y into X for if $Y^1 \subseteq Y$, $X^1 = \{v \in X: v \perp y$ for some $y \in Y^1\}$, and $|X^1| < |Y^1|$, then $W_2(H \cup Y^1) > W_2(H)$.

We then assume that the elements of Y are labeled so that $Y = \{y_1, y_2, \ldots, y_s\}$, $s \leq r$, and $y_i \perp H_i$ in (X, Y) for each $i \leq s$. We then choose vertices $a_1, a_2, \ldots, a_s$ from $H_1, H_2, \ldots, H_s$ so that $y_i \perp a_i$ in G for each $i \leq s$. From the inductive hypothesis, we conclude that for each $i \leq s$, the subgraph $H_i - a_i$ can be partitioned into $W_2(H_i) - 1$ complete subgraphs each of at most two vertices.

Since $s \geq 1$, we may partition for each i with $s + 1 \leq i \leq r$, the subgraph H_i into $W_2(H_i)$ complete subgraphs of at most two vertices. When combined with $\{y_1, a_1\}, \{y_2, a_2\}, \ldots, \{y_s, a_s\}$, the construction produces a partition of G into $W_2(G)$ complete subgraphs of at most two vertices.

Anderson [1] uses a similar argument to give an elementary proof of Tutte's factor theorem from Hall's matching theorem.

It is not true that $W_3(G) = P_3(G)$ for all graphs. An example of a poset X for which $W_3(X) < P_3(X)$ is $(\underline{3} + \underline{3}) + \overline{3}$.

6. An extension of Dilworth's embedding theorem. In this section we consider the structure of $\underline{2}^X$ in more detail in order to make an exact computation of $\mathrm{Dim}_k \underline{2}^X$.

Theorem 7. $\mathrm{Dim}_k \underline{2}^X = P_{k-1}(X)$ *for all* X.

Proof. Suppose $\mathrm{Dim}_k \underline{2}^X = t$ and let $F: \underline{2}^X \rightarrow \underline{k}^t$ be an embedding. For each $x \in X$ let $f_x: X \rightarrow \underline{2}$ be defined by $f_x(y) = 0$ if $y \leq x$ in X and $f_x(y) = 1$ otherwise. It follows that $f_x \in \underline{2}^X$ for every $x \in X$ and $f_x < f_y$ in $\underline{2}^X$

iff $x > y$ in X, i.e. the map $g: \hat{X} \to \underline{2}^X$ defined by $g(x) = f_x$ is an embedding.

For each $i \leq t$ let $X_i = \{x \in X: y < x \text{ or } yIx \text{ implies } F(f_x)(i) < F(f_y)(i)\}$. Then each X_i is a chain in X with $|X_i| \leq k$. Furthermore if $|X_i| = k$, then the least element in X_i is also the least element in X.

We now show that $X = X_1 \cup X_2 \cup \cdots \cup X_t$. Suppose on the contrary that there exists $x \in X$ with $x \notin X_1 \cup X_2 \cup \cdots \cup X_t$. Then for each $i \leq t$, there exists a point $y \in X$ with $y \not\geq X$ but $F(f_x)(i) \geq F(f_y)(i)$. Let $\mathcal{C}$ be the collection of all subsets $A \subseteq X$ such that (1) $a \in A$ implies $a \not\geq x$ and (2) for every $i \leq t$, there exists $a \in A$ with $F(f_x)(i) \geq F(f_a)(i)$. Now among the sets in $\mathcal{C}$, choose one set say A_0 with $|A_0|$ minimum. It follows that A_0 is an antichain and $|A_0| \geq 2$. Now define a function $f_0: X \to \underline{2}$ by $f_0(y) = 0$ if $y \leq a$ for some $a \in A_0$ and $f_0(y) = 1$ otherwise. It follows that $f_0 \in \underline{2}^X$ and $f_0 < f_a$ in $\underline{2}^X$ for every $a \in A_0$. Furthermore $f_0 \not\leq f_x$ in $\underline{2}^X$ since $f_0(x) = 1$ and $f_x(x) = 0$. Since F is an embedding of $\underline{2}^X$ in $\underline{k}^t$, there exist $i \leq t$ with $F(f_0)(i) > F(f_x)(i)$ and thus $F(f_a)(i) > F(f_x)(i)$ for every $a \in A_0$. The contradiction shows that $X = X_1 \cup X_2 \cup \cdots \cup X_t$.

If X has no least element, then $|X_i| \leq k - 1$ for all $i \leq t$ and thus $P_{k-1}(X) \leq t$. If X has a least element x, remove x from each chain in which it appears and let the resulting chains be $Y_1, Y_2, \ldots, Y_t$. If $|Y_i| \leq k - 2$ for some $i \leq t$, then we conclude that $P_{k-1}(X) \leq t$ since

$$X = Y_1 \cup Y_2 \cup \cdots \cup (Y_i \cup \{x\}) \cup \cdots \cup Y_t.$$

If $|Y_i| = k - 1$ for every $i \leq t$, then $F(f_x)(i) = k - 1$ for every $i \leq t$. Define $h: X \to \underline{2}$ by $h(y) = 1$ for all $y \in X$. Then $h > f_x$ in $\underline{2}^X$ but $F(f_x) \geq F(h)$ in $\underline{k}^t$. The contradiction completes the proof.

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