Journal Title: Journal of graph theory

Volume: 18
Issue: 6
Month/Year: 1994
Pages: 569-584

Article Author: Kierstead

Article Title: Planar graph coloring with an uncooperative partner

William Trotter (wt48)
School of Mathematics
Georgia Tech
Atlanta, GA 30332

Faculty
Math

Planar Graph Coloring with an Uncooperative Partner

H. A. Kierstead
DEPARTMENT OF MATHEMATICS
ARIZONA STATE UNIVERSITY
TEMPE, ARIZONA

W. T. Trotter
BELL COMMUNICATIONS RESEARCH
MORRISTOWN, NEW JERSEY
DEPARTMENT OF MATHEMATICS
ARIZONA STATE UNIVERSITY
TEMPE, ARIZONA

ABSTRACT

We show that the game chromatic number of a planar graph is at most 33. More generally, there exists a function $f: \mathbb{N} \to \mathbb{N}$ so that for each $n \in \mathbb{N}$, if a graph does not contain a homeomorph of K_n, then its game chromatic number is at most $f(n)$. In particular, the game chromatic number of a graph is bounded in terms of its genus. Our proof is motivated by the concept of p-arrangeability, which was first introduced by Guantao and Schelp in a Ramsey theoretic setting.

1. INTRODUCTION

Let $\mathbf{G} = (V, E)$ be a finite graph, and let X be a set whose elements will be referred to as *colors*. A function $c: V \to X$ is called a *proper coloring* (or just *coloring* for short) if $c(x) \neq c(y)$ whenever x and y are distinct nodes from V with $xy \in E$. If $|\{c(x): x \in V\}| = t$, the coloring c is also called a t-*coloring*. The *chromatic number* of $\mathbf{G}$, denoted $\chi(\mathbf{G})$, is the least positive integer t for which there exists a coloring c of $\mathbf{G}$ using a set X with $|X| = t$ as the set of colors.

In this paper, we will be concerned primarily with planar graphs. Because it is important to the spirit of the results that follow, we note that there is an elementary (and very fast) algorithm for coloring a planar graph with

Journal of Graph Theory, Vol. 18, No. 6, 569–584 (1994)
 CCC 0364-9024/94/060569-16

6 colors. By Euler's formula, a planar graph always has a node of degree at most 5. Given a graph $\mathbf{G} = (V, E)$ with n nodes, we can then label the nodes $x_1, x_2, \ldots, x_n$ so that for each $i = 2, 3, \ldots, n$, there are at most 5 neighbors of x_i in the set $\{x_j: 1 \leq j < i\}$. The graph can then be 6-colored by applying First-Fit to the nodes in the order of their subscripts in this labeling, i.e., a node is colored with the least positive integer distinct from the colors given to those neighbors that precede it in the labeling.

We now consider a modified graph coloring problem posed as a two-person game, with one person (Alice) trying to color a graph and the other (Bob) trying to prevent this from happening. Let $\mathbf{G} = (V, E)$ be a graph, let t be a positive integer, and let X be a set of colors with $|X| = t$. Alice and Bob compete in a two-person game lasting at most $n = |V|$ *moves*. They alternate turns, with Alice having the first move. A move consists of selecting a previously uncolored node x and assigning it a color from X distinct from the colors assigned previously (by either player) to neighbors of x. If after n moves, the graph is colored, Alice is the winner. Bob wins if an impass is reached before all nodes in the graph are colored, i.e., for every uncolored node x and every color α from X, x is adjacent to a node having color α. The *game chromatic number* of a graph $\mathbf{G} = (V, E)$, denoted $\chi_g(\mathbf{G})$, is the least t for which Alice has a winning strategy. This parameter is well defined, since Alice always wins when $t = |V|$.

Example. Consider the planar graph shown in Figure 1. This graph has game chromatic number 6. To see that the game chromatic number is at least 6, here is a winning strategy for Bob if the set X of colors is $\{1, 2, 3, 4, 5\}$. Note that for each $j = 1, 2, \ldots, 6$, the two-element set $\{a_j, b_j\}$ is a dominating set, i.e., every other node in the graph is adjacent to at least one of these two nodes. Each time Alice colors a node from $\{a_j, b_j\}$, say

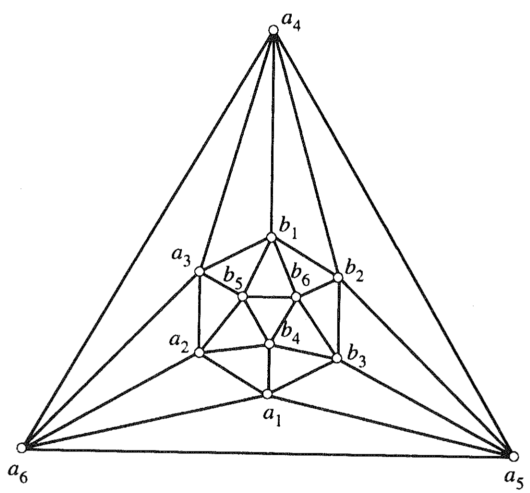

FIGURE 1

with color α, Bob responds by assigning color α to the other node in this set. It follows that α cannot be used by either player to color any other node in the graph. We leave it as an exercise to show that the game chromatic number is at most 6.

The *game chromatic* number of a family $\mathcal{F}$ of graphs, denoted $\chi(\mathcal{F})$, is then defined to be $\max\{\chi_g(\mathrm{G}): \mathrm{G} \in \mathcal{F}\}$, provided this value is finite; otherwise, we say that $\chi_g(\mathcal{F})$ is infinite.

The concept of game chromatic number was introduced by Bodlaender [1], who showed that the game chromatic number of the family of trees is at least 4 and at most 5. In [6], Faigle, Kern, Kierstead, and Trotter show that the game chromatic number of the family of trees is 4. In this paper, it is also shown that the family of bipartite graphs has infinite game chromatic number.

With these remarks as background, we can now state the principal result of this paper.

1.1 Theorem. The game chromatic number of the family of planar graphs is at most 33. ∎

Furthermore, we will produce a very fast procedure for implementing the winning strategy. As an added bonus, we obtain the following more general result.

1.2 Theorem. There exists a function $f: \mathbb{N} \to \mathbb{N}$ so that for each $n \in \mathbb{N}$, if a graph does not contain a homeomorph of K_n, then its game chromatic number is at most $f(n)$. ∎

It follows from Theorem 1.2 that there exists a function $g: \mathbb{N} \to \mathbb{N}$ so that $\mathbf{G}$ is a graph of genus n; then the game chromatic number of $\mathbf{G}$ is at most $g(n)$.

2. ARRANGEABILITY AND RAMSEY THEORY

Let $\mathbf{G} = (V, E)$ be a graph and let L be a linear order on the node set V. For each node $x \in V$, we define the *back degree of* x *relative to* L as $|\{y \in V: xy \in E \text{ and } x > y \text{ in } L\}|$. The *back degree* of L is then the maximum back degree of the nodes relative to L. The graph $\mathbf{G} = (V, E)$ is said to be k-*degenerate* if there is a linear order L on V that has back degree at most k. If $\mathbf{G}$ is k-degenerate, then $\chi(\mathbf{G}) \leq k + 1$, since First-Fit will use at most $k + 1$ colors when the nodes are processed in the linear order that witnesses that the graph is k-degenerate.

Again, let L be a linear order on the node set V of a graph $\mathbf{G} = (V, E)$, and let $x \in V$. We define the *arrangeability of* x *relative to* L as $|\{y \in V: y \leq x$ in L and there is some $z \in V$ with $yz \in E$, $xz \in E$ and $x < z$ in $L\}|$. In Figure 2, we illustrate a linear order L on the nodes of a graph. In this