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On the fractional dimension of partially ordered sets

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Abstract

We use a variety of combinatorial techniques to prove several theorems concerning fractional dimension of partially ordered sets. In particular, we settle a conjecture of Brightwell and Scheinerman by showing that the fractional dimension of a poset is never more than the maximum degree plus one. Furthermore, when the maximum degree k is at least two, we show that equality holds if and only if one of the components of the poset is isomorphic to S_{k+1} , the 'standard example' of a $k+1$ -dimensional poset. When $w \geq 3$, the fractional dimension of a poset P of width w is less than w unless P contains S_w . If P is a poset containing an antichain A and at most n other points, where $n \geq 3$, we show that the fractional dimension of P is less than n unless P contains S_n . If P contains an antichain A such that all antichains disjoint from A have size at most $w \geq 4$, then the fractional dimension of P is at most $2w$, and this bound is best possible.

Keywords: Partially poset; Dimension; Fractional dimension; Degree

1. Introduction

In this paper, we consider a *partially ordered set* P as a pair (X, P) , and refer to X as the *ground set* and P as the *partial order*. We find it convenient to use the short form *poset* for a partially ordered set, and we use the term *subposet* to refer to a poset induced by a subset of the ground set.

Let $P = (X, P)$ be a poset and let $\mathcal{F} = \{M_1, \dots, M_t\}$ be a multiset of linear extensions of P . Brightwell and Scheinerman [3] call \mathcal{F} a *k-fold realizer* of P if for each incomparable pair (x, y) , there are at least k linear extensions in \mathcal{F} which reverse the pair (x, y) , i.e., $|\{i: 1 \leq i \leq t, x > y \text{ in } M_i\}| \geq k$. The *fractional dimension* of P , denoted by $\text{fdim}(P)$, is then defined in [3] as the least real number $q \geq 1$ for which there exists

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a k -fold realizer $\mathcal{F} = \{M_1, \dots, M_k\}$ of P so that $k/t \geq 1/q$ (it is easily verified that the least upper bound of such real numbers q is indeed attained). Using this terminology, the *dimension* of P , denoted by $\dim(P)$, is just the least t for which there exists a t -fold realizer of P . It follows immediately that $\text{fdim}(P) \leq \dim(P)$, for every poset P . In this paper, we will find it convenient to use the following probabilistic interpretation of this concept. Let $P = (X, P)$ be a poset and let $\mathcal{F} = \{M_1, \dots, M_t\}$ be a multiset of linear extensions of P . We consider the linear extensions of \mathcal{F} as outcomes in a uniform sample space. For an incomparable pair (x, y) , the probability that x is over y in \mathcal{F} is given by

$$\text{Prob}_{\mathcal{F}}[x > y] = \frac{1}{t} |\{i: 1 \leq i \leq t, x > y \text{ in } M_i\}|.$$

The fractional dimension of P is then the least rational number $q \geq 1$ so that there exists a multiset $\mathcal{F} = \{M_1, \dots, M_t\}$ of linear extensions of P with $\text{Prob}_{\mathcal{F}}[x > y] \geq 1/q$, for every incomparable pair (x, y) .

For each $n \geq 3$, the height two poset containing n minimal elements $\{x_1, x_2, \dots, x_n\}$ and n maximal elements $\{y_1, y_2, \dots, y_n\}$ with $x_i < y_j$ if and only if $i \neq j$, for all $i, j = 1, 2, \dots, n$ will be denoted by S_n . The poset S_n is known as the *standard example* of a n -dimensional poset. As noted in [3], $\text{fdim}(S_n) = \dim(S_n) = n$.

This paper is organized as follows. In Section 2, we introduce some terminology necessary for our theorems and review facts about lexicographic sums. In Section 3, we formulate the results relating the fractional dimension of a poset to its degree. In Section 4, we present a key technical lemma involving the algorithmic transformation of linear orders into linear extensions and present the proof for the main theorem of Section 3. In Section 5, we present the forbidden subposet characterization of the degree inequality. In Sections 6 and 7, we derive two other forbidden subposet characterizations of inequalities for fractional dimension. In both cases equality is only possible when P contains a 'full dimensional' standard example as a subposet. In Section 8, we present an inequality relating the fractional dimension of a poset $P = (X, P)$ to the width of the subposet induced by $X - A$, where A is an antichain. The bounds obtained in this case are slightly stronger than the corresponding bounds for dimension. We also prove that our bounds are best possible. In Section 9, we characterize Hiraguchi's inequality for fractional dimension. Finally, in Section 10, we propose some new problems for fractional dimension.

2. Notation and terminology

Let $P = (X, P)$ be a poset. We denote the set of incomparable pairs of P by $\text{inc}(P)$. Now let $\mathcal{F} = \{M_1, M_2, \dots, M_t\}$ be a nonempty multiset of linear orders on X . We call \mathcal{F} a *multirealizer* of P if $P = \bigcap \mathcal{F}$, i.e., each M_i is a linear extension of P and $\text{prob}_{\mathcal{F}}[x > y] > 0$, for every $(x, y) \in \text{inc}(P)$.

A poset S is *decomposable* if there is a linear extension L of S such that L is the union of two posets U and V such that U and V are incomparable in L . Two posets U and V are *incomparable* if there is no element x in U such that $x < y$ for some y in V , and vice versa. The meaning is clear and $U(x)$. Two posets U and V are *comparable* if there is an element x in U such that $x < y$ for some y in V , and vice versa. The meaning is clear and $U(x)$.

(1) $\dim(S) = \text{max}\{k: S \text{ has a } k\text{-fold realizer}\}$

(2) $\text{fdim}(S) = \text{min}\{q: S \text{ has a } q\text{-fold realizer}\}$

Both dimension and fractional dimension are additive over disjoint posets.

Let $P = (X, P)$ be a poset. A *linear extension* of P is a linear order L on X such that L is an extension of P .

Let $P = (X, P)$ be a poset. A *multirealizer* of P is a multiset \mathcal{F} of linear orders on X such that $P = \bigcap \mathcal{F}$.

Let $P = (X, P)$ be a poset. A *realizer* of P is a set \mathcal{R} of linear orders on X such that $P = \bigcap \mathcal{R}$.

Let $P = (X, P)$ be a poset. A *fractional realizer* of P is a multiset \mathcal{F} of linear orders on X such that $P = \bigcap \mathcal{F}$ and $\text{prob}_{\mathcal{F}}[x > y] > 0$, for every $(x, y) \in \text{inc}(P)$.

Let $P = (X, P)$ be a poset. A *width* of P is the size of a largest antichain in P .

Let $P = (X, P)$ be a poset. A *height* of P is the length of a longest chain in P .

Let $P = (X, P)$ be a poset. A *chain* of P is a subset C of X such that C is a chain in P .

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