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The Dimension of Suborders of the Boolean Lattice

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Abstract. We consider the order dimension of suborders of the Boolean lattice $\mathbf{B}_n$. In particular we show that the suborder consisting of the middle two levels of $\mathbf{B}_n$ has dimension at most $6\log_3 n$. More generally, we show that the suborder consisting of levels s and $s+k$ of $\mathbf{B}_n$ has dimension $O(k^2 \log n)$.

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1. Introduction

For any positive integer n, let $[n] = \{1, 2, \ldots, n\}$, let $\mathcal{B}_n$ be the collection of subsets of $[n]$, and let $\mathbf{B}_n = (\mathcal{B}_n, \subseteq)$ denote the Boolean lattice, where the subsets of $[n]$ are ordered by inclusion. For a finite set A, let $C(A, k)$ denote the collection of k-element subsets of A. For integers n, s and t with $0 \leqslant s < t \leqslant n$, let $\mathbf{B}_n(s,t)$

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denote the restriction of $\mathcal{B}_n$ to $C([n], s) \cup C([n], t)$. Finally, let $\dim(s, t; n)$ denote the (order) dimension of $\mathcal{B}_n(s, t)$. We refer the reader to the monograph [7] for additional background material on dimension theory.

The function $\dim(s, t; n)$ was first studied by Dushnik [1] in 1950, but estimates for the function are surprisingly poor, except in the case $s = 1$. In this case, Dushnik noted the following useful reformulation of the problem.

PROPOSITION 1.1. *For all positive integers t and n, $1 < t < n$, $\dim(1, t; n)$ is the least positive integer d for which there exists a set Σ of d linear orderings of $[n]$ such that for all $X \in C([n], t)$ and all $y \in [n] - X$, there exists $L \in \Sigma$ such that in L, y is greater than every element of X.*

With the aid of Proposition 1.1, Dushnik [1] proved the following result, establishing the exact value for $\dim(1, t; n)$ when $t \geqslant 2\sqrt{n} - 2$.

THEOREM 1.2 [1]. *Let n and t be positive integers with $n \geqslant 4$ and $2\sqrt{n} - 2 \leqslant t \leqslant n - 1$. Then let j be the unique integer with $2 \leqslant j \leqslant \sqrt{n}$ for which*

$$\left\lfloor \frac{n - 2j + j^2}{j} \right\rfloor \leqslant t < \left\lfloor \frac{n - 2(j-1) + (j-1)^2}{j - 1} \right\rfloor.$$

Then

$$\dim(1, t; n) = n - j + 1.$$

In the remainder of this paper, we will discuss *estimates* for the dimension of ordered sets. For this reason, we will omit "floors" and "ceilings" from expressions which only have meaning for integers.

For fixed t, Spencer [6] established the asymptotic behavior of $\dim(1, t; n)$.

THEOREM 1.2 [6]. *For fixed t,*

$$\dim(1, t; n) = \Theta(\log \log n).$$

The following elementary result is an exercise in [7] and follows easily from Dushnik's proof of Theorem 1.2.

PROPOSITION 1.4. *For all positive integers t and n with $t^2 \leqslant n$,*

$$t^2/4 < \dim(1, t; n).$$

In view of Proposition 1.4, the following result of Füredi and Kahn [4] establishes the value of $\dim(1, t; n)$ within a multiplicative factor of order $\log t$, if $t = \Omega(n^{\epsilon})$. The proof is simply a matter of taking d linear orderings of $[n]$, uniformly at random from the set of all possible linear orderings, and noting that the probability that these do not form a family Σ as in Proposition 1.1 tends to 0.

PROPOSITION 1.5 [4]. *For all positive inte integer satisfying*

$$n\binom{n-1}{t}\left(\frac{t}{t+1}\right)^d < 1,$$

then $\dim(1, t; n) \leqslant d$. *In particular,*

$$\dim(1, t; n) \leqslant (t+1)^2 \log n.$$

Determining $\dim(1, t; n)$ for t a small grow open problem. Moreover, until recently, ve Here are two well known trivial bounds.

PROPOSITION 1.6. *For all positive intege*

$$\dim(s', t'; n') \leqslant \dim(s, t; n).$$

PROPOSITION 1.7. *For all positive intege*

$$\dim(s - r, t - r; n - r) \leqslant \dim(s, t; n).$$

The next two results are given by Hurlbert,

THEOREM 1.8 [5]. *For each positive inte*

$$\dim(2, n - 2; n) = n - 1.$$

THEOREM 1.9 [5]. *For each positive inte*

$$\dim(2, n - 3; n) = n - 2.$$

In fact, it is shown in [5] that if $2\sqrt{n} < t$ $j - 2 + (n-1)/j$, for some positive integer

While preparing this manuscript, we ha the following result.

THEOREM 1.10 [2]. *For each integer k*

$$\dim(k, n - k; n) = n - 2.$$

In this note, we provide the following up parameters $\dim(1, 2(t - s); n)$ and $t - s$.

PROPOSITION 1.5 [4]. *For all positive integers t, n, with $t < n$, if d is a positive integer satisfying*

$$n\binom{n-1}{t}\left(\frac{t}{t+1}\right)^{d} < 1,$$

then $\dim(1,t;n) \leqslant d$. *In particular,*

$$\dim(1,t;n) \leqslant (t+1)^2 \log n.$$

Determining $\dim(1,t;n)$ for t a small growing function of n remains an intriguing open problem. Moreover, until recently, very little was known for the case $s > 1$. Here are two well known trivial bounds.

PROPOSITION 1.6. *For all positive integers* $s \leqslant s' < t' \leqslant t \leqslant n' \leqslant n$,

$$\dim(s',t';n') \leqslant \dim(s,t;n).$$

PROPOSITION 1.7. *For all positive integers* $r < s < t < n$,

$$\dim(s-r,t-r;n-r) \leqslant \dim(s,t;n).$$

The next two results are given by Hurlbert, Kostochka and Talysheva in [5].

THEOREM 1.8 [5]. *For each positive integer n with* $n \geqslant 5$,

$$\dim(2,n-2;n) = n-1.$$

THEOREM 1.9 [5]. *For each positive integer n with* $n \geqslant 6$,

$$\dim(2,n-3;n) = n-2.$$

In fact, it is shown in [5] that if $2\sqrt{n} < t < n-2$ and t is not an integer of the form $j-2+(n-1)/j$, for some positive integer j, then $\dim(2,t;n) = \dim(1,t-1;n-1)$.

While preparing this manuscript, we have just learned that Füredi [2] has proven the following result.

THEOREM 1.10 [2]. *For each integer* $k \geqslant 3$, *there exists* n_0 *so that if* $n > n_0$, *then*

$$\dim(k,n-k;n) = n-2.$$

In this note, we provide the following upper bound on $\dim(s,t;n)$ in terms of the parameters $\dim(1,2(t-s);n)$ and $t-s$.