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The Dimension of Suborders of the Boolean Lattice

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Abstract. We consider the order dimension of suborders of the Boolean lattice \mathcal{B}_n . In particular we show that the suborder consisting of the middle two levels of \mathcal{B}_n has dimension at most $6 \log_3 n$. More generally, we show that the suborder consisting of levels s and $s+k$ of \mathcal{B}_n has dimension $O(k^2 \log n)$.

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1. Introduction

For any positive integer n , let $[n] = \{1, 2, \dots, n\}$, let \mathcal{B}_n be the collection of subsets of $[n]$, and let $\mathcal{B}_n = (\mathcal{B}_n, \subseteq)$ denote the Boolean lattice, where the subsets of $[n]$ are ordered by inclusion. For a finite set A , let $C(A, k)$ denote the collection of k -element subsets of A . For integers n, s and t with $0 \leq s < t \leq n$, let $\mathcal{B}_n(s, t)$

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denote the restriction of B_n to $C([n], s) \cup C([n], t)$. Finally, let $\dim(s, t; n)$ denote the (order) dimension of $B_n(s, t)$. We refer the reader to the monograph [7] for additional background material on dimension theory.

The function $\dim(s, t; n)$ was first studied by Dushnik [1] in 1950, but estimates for the function are surprisingly poor, except in the case $s = 1$. In this case, Dushnik noted the following useful reformulation of the problem.

PROPOSITION 1.1. For all positive integers t and n , $1 < t < n$, $\dim(1, t; n)$ is the least positive integer d for which there exists a set Z of d linear orderings of $[n]$ such that for all $X \in C([n], t)$ and all $y \in [n] - X$, there exists $L \in Z$ such that in L , y is greater than every element of X .

With the aid of Proposition 1.1, Dushnik [1] proved the following result, establishing the exact value for $\dim(1, t; n)$ when $t \geq 2\sqrt{n} - 2$.

THEOREM 1.2 [1]. Let n and t be positive integers with $n \geq 4$ and $2\sqrt{n} - 2 \leq t < n - 1$. Then let j be the unique integer with $2 \leq j \leq \sqrt{n}$ for which

$$\left\lfloor \frac{j}{n - 2j + j^2} \right\rfloor \leq t < \left\lfloor \frac{j - 1}{n - 2(j - 1) + (j - 1)^2} \right\rfloor.$$

Then

$$\dim(1, t; n) = n - j + 1.$$

In the remainder of this paper, we will discuss *estimates* for the dimension of ordered sets. For this reason, we will omit "floors" and "ceilings" from expressions which only have meaning for integers. For fixed t , Spencer [6] established the asymptotic behavior of $\dim(1, t; n)$.

THEOREM 1.2 [6]. For fixed t , $\dim(1, t; n) = \Theta(\log \log n)$.

The following elementary result is an exercise in [7] and follows easily from Dushnik's proof of Theorem 1.2.

PROPOSITION 1.4. For all positive integers t and n with $t^2 \leq n$,

$$t^2/4 > \dim(1, t; n).$$

In view of Proposition 1.4, the following result of Füredi and Kahn [4] establishes the value of $\dim(1, t; n)$ within a multiplicative factor of order $\log t$, if $t = \Omega(n^{\epsilon})$. The proof is simply a matter of taking d linear orderings of $[n]$, uniformly at random from the set of all possible linear orderings, and noting that the probability that these do not form a family Z as in Proposition 1.1 tends to 0.

PROPOSITION 1.5 [4]. For all positive integer satisfying

$$n \binom{n-1}{t} \left(\frac{t}{t+1} \right)^d > 1,$$

then $\dim(1, t; n) \leq d$. In particular,

$$\dim(1, t; n) \leq (t+1)^2 \log n.$$

Determining $\dim(1, t; n)$ for t a small growth open problem. Moreover, until recently, there are two well known trivial bounds.

PROPOSITION 1.6. For all positive integers

$$\dim(s', t'; n') \leq \dim(s, t; n).$$

PROPOSITION 1.7. For all positive integers

$$\dim(s - r, t - r; n - r) \leq \dim(s, t; n).$$

The next two results are given by Huribert,

THEOREM 1.8 [5]. For each positive integer

$$\dim(2, n - 2; n) = n - 1.$$

THEOREM 1.9 [5]. For each positive integer

$$\dim(2, n - 3; n) = n - 2.$$

In fact, it is shown in [5] that if $2\sqrt{n} < t - 2 + (n - 1)/j$, for some positive integer j , while preparing this manuscript, we have the following result.

THEOREM 1.10 [2]. For each integer k ,

$$\dim(k, n - k; n) = n - 2.$$

In this note, we provide the following parameters $\dim(1, 2(t - s); n)$ and $t - s$.

$UC([n], t)$. Finally, let $\dim(s, t; n)$ denote the dimension of the suborder $UC([n], t)$. For additional information, see the reader to the monograph [7] for additional information.

studied by Dushnik [1] in 1950, but estimates were given except in the case $s = 1$. In this case, Dushnik [1] gave an estimate of the problem.

For integers t and n , $1 < t < n$, $\dim(1, t; n)$ is the dimension of the suborder $UC([n], t)$. It is known that there exists a set Σ of d linear orderings of $[n]$ such that for any $X \in [n] - X$, there exists $L \in \Sigma$ such that in L , X is a suborder.

Dushnik [1] proved the following result, establishing an upper bound $\dim(1, t; n) \leq 2\sqrt{n} - 2$.

For positive integers with $n \geq 4$ and $2\sqrt{n} - 2 \leq t \leq n$, with $2 \leq j \leq \sqrt{n}$ for which

$$\left\lfloor \frac{(j-1) + (j-1)^2}{-1} \right\rfloor.$$

Discuss estimates for the dimension of ordered suborders and "ceilings" from expressions which

the asymptotic behavior of $\dim(1, t; n)$.

Exercise in [7] and follows easily from Dushnik's result.

For integers t and n with $t^2 \leq n$,

A result of Füredi and Kahn [4] establishes an asymptotic multiplicative factor of order $\log t$, if $t = \Omega(n^\epsilon)$. For linear orderings of $[n]$, uniformly at random, and noting that the probability that these suborders are linear tends to 0.

PROPOSITION 1.5 [4]. For all positive integers t, n , with $t < n$, if d is a positive integer satisfying

$$n \binom{n-1}{t} \left(\frac{t}{t+1} \right)^d < 1,$$

then $\dim(1, t; n) \leq d$. In particular,

$$\dim(1, t; n) \leq (t+1)^2 \log n.$$

Determining $\dim(1, t; n)$ for t a small growing function of n remains an intriguing open problem. Moreover, until recently, very little was known for the case $s > 1$. Here are two well known trivial bounds.

PROPOSITION 1.6. For all positive integers $s \leq s' < t' \leq t \leq n' \leq n$,

$$\dim(s', t'; n') \leq \dim(s, t; n).$$

PROPOSITION 1.7. For all positive integers $r < s < t < n$,

$$\dim(s-r, t-r; n-r) \leq \dim(s, t; n).$$

The next two results are given by Hurlbert, Kostochka and Talysheva in [5].

THEOREM 1.8 [5]. For each positive integer n with $n \geq 5$,

$$\dim(2, n-2; n) = n-1.$$

THEOREM 1.9 [5]. For each positive integer n with $n \geq 6$,

$$\dim(2, n-3; n) = n-2.$$

In fact, it is shown in [5] that if $2\sqrt{n} < t < n-2$ and t is not an integer of the form $j-2 + (n-1)/j$, for some positive integer j , then $\dim(2, t; n) = \dim(1, t-1; n-1)$.

While preparing this manuscript, we have just learned that Füredi [2] has proven the following result.

THEOREM 1.10 [2]. For each integer $k \geq 3$, there exists n_0 so that if $n > n_0$, then

$$\dim(k, n-k; n) = n-2.$$

In this note, we provide the following upper bound on $\dim(s, t; n)$ in terms of the parameters $\dim(1, 2(t-s); n)$ and $t-s$.

