Journal Title: Order

Volume: 12
Issue: 4
Month/Year: 1995
Pages: 327-349

Article Author: Brightwell

Article Title: Balancing pairs and the cross product conjecture

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Order **12**: 327–349, 1995.

Balancing Pairs and the Cross Product Conjecture*

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Communicated by I. Rival

(Received:: 20 June 1995; accepted: 31 August 1995)

Abstract. In a finite partially ordered set, $\mathrm{Prob}(x > y)$ denotes the proportion of linear extensions in which element x appears above element y. In 1969, S. S. Kislitsyn conjectured that in every finite poset which is not a chain, there exists a pair (x, y) for which $1/3 \leqslant \mathrm{Prob}(x > y) \leqslant 2/3$. In 1984, J. Kahn and M. Saks showed that there exists a pair (x, y) with $3/11 < \mathrm{Prob}(x > y) < 8/11$, but the full 1/3–2/3 conjecture remains open and has been listed among ORDER's featured unsolved problems for more than 10 years.

In this paper, we show that there exists a pair (x, y) for which $(5 - \sqrt{5})/10 \leqslant \mathrm{Prob}(x > y) \leqslant (5 + \sqrt{5})/10$. The proof depends on an application of the Ahlswede–Daykin inequality to prove a special case of a conjecture which we call the Cross Product Conjecture. Our proof also requires the full force of the Kahn–Saks approach – in particular, it requires the Alexandrov–Fenchel inequalities for mixed volumes.

We extend our result on balancing pairs to a class of countably infinite partially ordered sets where the 1/3–2/3 conjecture is *false*, and our bound is best possible. Finally, we obtain improved bounds for the time required to sort using comparisons in the presence of partial information.

Mathematics Subject Classifications (1991). 06A07, 06A10.

Key words. Partially ordered set, linear extension, balancing pairs, cross-product conjecture, Ahlswede–Daykin inequality, sorting.

* An extended abstract of an earlier version of this paper appears as [6]. The results here are much stronger than in [6], and this paper has been written so as to overlap as little as possible with that version.

1. Introduction

Given a finite partially ordered set (poset) P, let $\Lambda(P)$ denote the set of linear extensions of P, and let $L(P) = |\Lambda(P)|$. For a pair x, y of distinct elements of P, let $\text{Prob}(x > y)$ denote the number of linear extensions of P in which $x > y$, divided by $L(P)$. Thus $\text{Prob}(x > y)$ is the proportion of linear extensions in which x is above y. The probabilistic notation is of course quite natural, corresponding to making $\Lambda(P)$ into a probability space with each linear extension equally likely. If $x < y$ in P, then $\text{Prob}(x > y) = 0$, while $\text{Prob}(x > y) = 1$ if $x > y$ in P. On the other hand, if x and y are incomparable in P, then $0 < \text{Prob}(x > y) < 1$. In 1969, S. S. Kislitsyn [15] made the following conjecture, which remains one of the most intriguing problems in the combinatorial theory of posets.

CONJECTURE 1.1. *If P is a finite poset which is not a chain, then there exists an incomparable pair $x, y \in P$ so that*

$$1/3 \leqslant \text{Prob}(x > y) \leqslant 2/3.$$

Conjecture 1.1 was also made independently by both M. Fredman and N. Linial, and many papers on this subject attribute the conjecture to them. It is now known as the 1/3–2/3 conjecture. If true, the conjecture would be best possible, as shown by the poset with three elements and one comparable pair.

The 1/3–2/3 conjecture has been proved for several special classes of posets. Linial [17] showed that the conjecture holds for width two posets, and P. Fishburn, W. G. Gehrlein and W. T. Trotter [8] showed that it holds for height two posets. G. Brightwell [3] showed that it holds for semiorders, and Brightwell and C. D. Wright [5] verified it for posets in which every element is incomparable with at most five others.

Following Kahn and Saks, for a finite partially ordered set P, we let $\delta(P)$ denote the largest positive real number so that there exists a pair (x, y) of distinct points from P with $\delta(P) \leqslant \text{Prob}(x > y) \leqslant 1 - \delta(P)$. We may then set δ_0 to be the infimum of $\delta(P)$, taken over all finite P which are not chains. With this notation, the 1/3–2/3 conjecture is just the assertion that $\delta_0 \geqslant 1/3$. However, to the best of our knowledge, there is no entirely elementary proof that $\delta_0 > 0$.

The first major breakthrough in this area came in 1984, when Kahn and Saks [13] used the Alexandrov–Fenchel inequalities for mixed volumes to prove the following result.

THEOREM 1.2. *If P is a finite poset which is not a chain, then there exists an incomparable pair $x, y \in P$ so that*

$$3/11 < \text{Prob}(x > y) < 8/11.$$

Thus, $\delta_0 > 3/11 \simeq 0.2727$. □

Our main result is the following inequality
rem 1.2.

THEOREM 1.3. *If P is a finite poset which
incomparable pair $x, y \in P$ so that*

$$(5 - \sqrt{5})/10 \leqslant \text{Prob}(x > y) \leqslant (5 + \sqrt{}$$

Thus, $\delta_0 \geqslant (5 - \sqrt{5})/10 \simeq 0.2764$.

The proof of Theorem 1.3 requires all the ma
for the proof of Theorem 1.2. In particular
inequalities for mixed volumes to prove tha

Our proof also requires a new inequality –
we call the *Cross Product Conjecture*. Alth
the Cross Product Conjecture in full general
purposes here. Even this case requires an
inequality – a deep and powerful combinat
wide range of applications to posets.

Numerically, Theorem 1.3 is only a mo
bound and leaves us far short of settling the
sense, our Theorem 1.3 is best possible. To
investigation to include countably infinite p
1/3–2/3 conjecture is false. Call a poset P
such that every element of P is incomparabl
has a connected incomparability graph, th
$[a, b] \equiv \{z\colon a \leqslant z \leqslant b\}$ is finite. Let $([a_n, b$
whose union is the ground-set of P. If the
intervals $[a_m, b_m]$, then for $n \geqslant m$ we can c
in the poset $P|[a_n, b_n]$. Brightwell [2] show
and is independent of the sequence of in
$\text{Prob}(x > y)$ in P to be this limit. This def
way to thin posets with disconnected incon

For a thin poset P, we again define $\delta(P)$
which there exists a pair $x, y \in P$ with $\delta($
let δ_0' be the infimum of $\delta(P)$ over all thin

As was discovered independently by Brig
poset Q with $\delta(Q) = (5 - \sqrt{5})/10$. This ex
poset Q has as its point set $X = \{x_i\colon i \in$
$j > i + 1$ in $\mathbb{Z}$. If we define the finite pose
of all points whose subscripts in absolute
exercise to show that

$$\lim_{n\to\infty} \text{Prob}(x_0 > x_1) = (5 - \sqrt{5})/10.$$