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CHAPTER 8

Partially Ordered Sets

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Contents

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Introduction

Interest in finite partially ordered sets has been heightened in recent years by a steady stream of theorems combining clever ad hoc arguments with powerful techniques from other areas of mathematics. In this chapter, we present a sampling of results exhibiting these characteristics. In those instances where we do not present a complete proof, we outline enough of the general contours of the argument to allow the reader to supply the missing details with little difficulty. We also outline anticipated research directions in the combinatorics of partially ordered sets, and we discuss briefly some of the most interesting open problems in this field.

Since this Handbook contains chapters on Extremal Set Theory and Enumeration, we have limited our discussion to results on general partially ordered sets. Still some difficult choices had to be made concerning results to be included – especially in view of our emphasis on proof techniques. West's survey articles (West 1982, 1985) offer more of a catalogue of theorems in the area and have extensive bibliographies. Also, we recommend the recent books by Anderson (1987), Fishburn (1986), Stanley (1986), and Trotter (1992) as well as the conference volumes (Rival 1982, 1985) for additional material on partially ordered sets and related topics.

1. Notation and terminology

Formally, a *partially ordered set* is a pair (X, P) where X is a set, and P is a reflexive, antisymmetric, and transitive binary relation on X. The set X is called the *ground set* and P is called a *partial order*. Throughout this chapter, we use the short form *poset* for a partially ordered set. Many researchers choose to drop the adjective "partially" and use *ordered set* to mean a poset. A poset (X, P) is *finite* if the ground set X is finite. In this chapter, we will be concerned primarily with finite posets.

In some settings, we find it convenient to use a single symbol such as $\mathbf{P}$ to denote a poset (X, P). This notation is particularly handy when both the ground set X and the partial order P remain fixed. In other settings, especially when we have several partial orders on the same ground set, we will use the ordered pair notation for posets.

The notations $(x, y) \in P$, xPy, $x \leqslant y$ in P, and $y \geqslant x$ in P are used interchangeably. The notation $x < y$ in P means $x \leqslant y$ in P and $x \neq y$. Distinct points x, y are *comparable* when either $x < y$ or $y < x$ in P. Otherwise, we say x and y are *incomparable* and write $x \| y$ in P. When using a single symbol like $\mathbf{P}$ for a poset, we will write $x < y$ in $\mathbf{P}$, $x \| y$ in $\mathbf{P}$, etc.

A poset $\mathbf{P} = (X, P)$ is a *chain* (also a *totally ordered set* or a *linearly ordered set*) if each pair of distinct points is comparable. We will use the symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote the *reals*, *rationals*, *integers* and *positive integers*, respectively. Each of these posets is a chain.

Dually, $\mathbf{P} = (X, P)$ is an *antichain* if each pair of distinct points is incomparable.

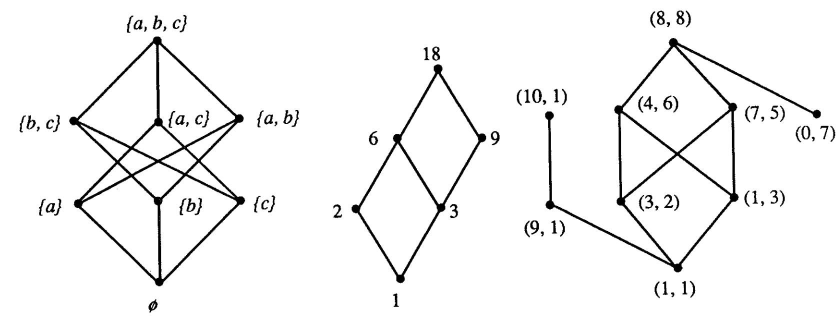


Figure 1.1.

If $Y \subset X$ and Q is the restriction of P to Y, then the poset $\mathbf{Q} = (Y, Q)$ is called a *subposet* of (X, P). A subset $Y \subset X$ is also called a *chain* (*antichain*) if the subposet (Y, Q) is a chain (antichain). The *height* of a poset is the maximum cardinality of a chain, and the *width* is the maximum cardinality of an antichain.

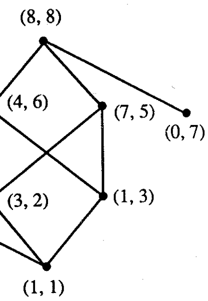
When $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ are posets, a map $f : X \to Y$ is called an *embedding* (of $\mathbf{P}$ into $\mathbf{Q}$) if $x_1 \leqslant x_2$ in $P \iff f(x_1) \leqslant f(x_2)$ in Q. An embedding $f : X \to Y$ is an *isomorphism* when $f(X) = Y$. In this chapter, we prefer not to distinguish between isomorphic posets and to write $\mathbf{P} = \mathbf{Q}$ to indicate that the two posets are isomorphic. Similarly we say that $\mathbf{P}$ is *contained* in $\mathbf{Q}$ (also $\mathbf{P}$ is a *subposet* of $\mathbf{Q}$) when there exists an embedding of $\mathbf{P}$ in $\mathbf{Q}$.

We say y *covers* x in P and write $x <: y$ in P when there is no z for which both $x < z$ and $z < y$ in P. The *cover graph* associated with the poset $\mathbf{P} = (X, P)$ is the graph $\mathbf{G} = (X, E)$ whose edge set E consists of the pairs xy for which $x <: y$ in P. A drawing of the cover graph $\mathbf{G} = (X, E)$ in the Euclidean plane is called a *Hasse diagram* (or *order diagram*) of the poset $\mathbf{P} = (X, P)$ if x is lower in the plane than y whenever $x <: y$ in P.

Here are some frequently encountered examples of posets. Any family of sets is partially ordered by set inclusion; a set of positive integers is partially ordered by division; and a subset of $\mathbb{R}^n$ is partially ordered by $(a_1, a_2, \ldots, a_n) \leqslant (b_1, b_2, \ldots, b_n) \iff a_i \leqslant b_i$ in $\mathbb{R}$ for $i = 1, 2, \ldots, n$. In fig. 1.1, we show particular instances of these examples. Each has height 4, and their respective widths are 3, 2, and 4.

If P and Q are partial orders on the same ground set X, Q is called an *extension* of P when $P \subseteq Q$. The partial order Q is called a *linear extension* of P if Q is an extension of P and (X, Q) is a chain.

When $\mathbf{P} = (X, P)$ is a poset, an element $x \in X$ is called a *maximal* (*minimal*) element if there is no $y \in X$ for which $x < y$ in P ($y < x$ in P). The set of maximal (minimal) elements is denoted $\mathrm{MAX}(X, P)$ ($\mathrm{MIN}(X, P)$). The subsets $\mathrm{MAX}(X, P)$



and $\mathrm{MIN}(X,P)$ always determine antichains, although neither may be as large as the width of (X,P).

When $Y \subset X$, the set $\{z \in X\colon y \leqslant z \text{ in } P \text{ for every } y \in Y\}$ is called the set of *upper bounds* for Y. Note that this set may be empty. When the set of upper bounds of Y is nonempty and has a least element, this unique point is called the *least upper bound* of Y and is denoted l.u.b.(Y). Dually, the *greatest lower bound* (if it exists) of Y is denoted g.l.b.(Y).

A poset $\mathbf{P} = (X,P)$ is called a *lattice* when each nonempty subset $Y \subset X$ has both a least upper bound and a greatest lower bound. When $\mathbf{P} = (X,P)$ is a lattice and $x, y \in X$, we write $x \vee y$ for l.u.b.$\{x,y\}$ and $x \wedge y$ for g.l.b.$\{x,y\}$. The binary operations $\vee$ (join) and $\wedge$ (meet) are commutative and associative. The lattice is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in X$.

When $\mathbf{P}$ and $\mathbf{Q}$ are posets, the *disjoint sum* of $\mathbf{P}$ and $\mathbf{Q}$, denoted $\mathbf{P}+\mathbf{Q}$, is obtained by taking the union of disjoint copies of the two posets with no comparabilities between the points in one and points in the other. A poset is *disconnected* if it is the disjoint sum of two proper subposets; otherwise it is *connected.* The maximal connected subposets of a disconnected poset are *components.*

The *cartesian product* of $\mathbf{P} = (X,P)$ and $\mathbf{Q} = (Y,Q)$, denoted $\mathbf{P} \times \mathbf{Q}$, consists of the ordered pairs (x,y) where $x \in X$ and $y \in Y$ with partial ordering $(x_1,y_1) \leqslant (x_2,y_2) \iff x_1 \leqslant x_2$ in $\mathbf{P}$ and $y_1 \leqslant y_2$ in $\mathbf{Q}$. The cartesian product of n copies of $\mathbf{P}$ is denoted $\mathbf{P}^n$.

Given posets $\mathbf{P} = (X,P)$ and $\mathbf{Q} = (Y,Q)$, a function $f\colon X \to Y$ is an *order preserving* (or *monotone*) map from $\mathbf{P}$ to $\mathbf{Q}$ if $x_1 \leqslant x_2$ in $P \Rightarrow f(x_1) \leqslant f(x_2)$ in Q. The set of all order preserving maps from $\mathbf{P}$ to $\mathbf{Q}$ is partially ordered by $f_1 \leqslant f_2 \iff f_1(x) \leqslant f_2(x)$ in Q for every $x \in X$. This poset is denoted $\mathbf{Q}^{\mathbf{P}}$.

Throughout the chapter, we use $\mathbf{k}$ to denote a k-element chain $0 < 1 < 2 < \cdots < k-1$. The poset $\mathbf{2}^n$ is isomorphic to the set of subsets of an n-element set partially ordered by inclusion. A poset $\mathbf{P}$ is a distributive lattice if and only if there is a poset $\mathbf{Q}$ so that $\mathbf{P}$ is isomorphic to $\mathbf{2}^{\mathbf{Q}}$ [see chapter 3 in Birkhoff (1973)].

When $\mathbf{P} = (X,P)$ is a poset and $\mathscr{F} = \{\mathbf{P}_x = (Y_x, Q_x)\colon x \in X\}$ is a family of posets indexed by the ground set of $\mathbf{P}$, the *lexicographic sum* of $\mathscr{F}$ over $\mathbf{P}$ is the poset whose ground set is $\{(x,y)\colon x \in X, y \in Y_x\}$. The partial ordering is defined by $(x_1,y_1) \leqslant (x_2,y_2) \iff (x_1 < x_2$ in $P)$ or $(x_1 = x_2$ and $y_1 \leqslant y_2$ in $Q_{x_1})$. A lexicographic sum is nontrivial if $|X| \geqslant 2$ and if at least one Y_x satisfies $|Y_x| \geqslant 2$. A poset $\mathbf{P}$ is *decomposable* if it is isomorphic to a nontrivial lexicographic sum; otherwise $\mathbf{P}$ is *indecomposable.* Note that the disjoint sum of two posets is a lexicographic sum over a 2-element antichain.

2. Dilworth's theorem and the Greene–Kleitman theorem

Dilworth's decomposition theorem (Dilworth 1950) has played an important role in motivating research in posets, as evidenced by results discussed in this section as well as in sections 3, 6, 7 and 8. Also, Dilworth's theorem surfaces in a variety of extremal problems (see, for example, Duffus et al. 1991). There are several elementary proofs; the one we present is patterned after Perles (1963).

Theorem 2.1. *If* $\mathbf{P} = (X,P)$ *is a poset of width n, then there exists a partition* $X = C_1 \cup C_2 \cup \cdots \cup C_n$ *where each* C_i *is a chain.*

Proof. We proceed by induction on $|X|$ and note that the result is trivial when $|X| = 1$. Assume validity when $|X| < k$ and consider a poset $\mathbf{P}$ with $|X| = k$. We may assume that the width n of $\mathbf{P}$ is larger than 1.

Choose $x \in \mathrm{MAX}(\mathbf{P})$ and $y \in \mathrm{MIN}(\mathbf{P})$ with $y \leqslant x$. Let $\mathbf{Q}$ be the poset obtained by removing x and y from $\mathbf{P}$. If the width of $\mathbf{Q}$ is less than n, then we can partition $\mathbf{Q}$ into fewer than n chains which together with the chain $\{x,y\}$ form a partition of X into (at most) n chains. So we may assume that $\mathbf{Q}$ has width n. Thus $y < x$ in $\mathbf{P}$. Choose an n-element antichain $A = \{a_1, a_2, \ldots, a_n\}$ in $\mathbf{Q}$.

Then let $U = \{u \in X\colon u \geqslant a_i \text{ for some } a_i \in A\}$ and $D = \{d \in X\colon d \leqslant a_j \text{ for some } a_j \in A\}$. Evidently $x \in U - D$ and $y \in D - U$. Thus there are chain partitions $U = C_1' \cup C_2' \cup \cdots \cup C_n'$ and $D = C_1'' \cup C_2'' \cup \cdots \cup C_n''$. We may label these chains so that $a_i \in C_i' \cap C_i''$ for $i = 1, 2, \ldots, n$. Then $C_i = C_i' \cup C_i''$ is a chain for each i and the desired partition is $X = C_1 \cup C_2 \cup \cdots \cup C_n$. □

In introductory combinatorics texts, Dilworth's theorem is grouped with other max–min theorems having a common theme: P. Hall's marriage theorem, the König–Egervary theorem, Menger's theorem, and the max flow–min cut theorem for network flows. This last result most clearly captures the linear programming core common to all. (See chapters 2 and 3 by Frank and Pulleyblank for additional material.)

Dilworth's theorem has a trivial dual version for antichains.

Theorem 2.2. *If* $\mathbf{P} = (X,P)$ *is a poset of height n, then there exists a partition* $X = A_1 \cup A_2 \cup \cdots \cup A_n$ *where each* A_i *is an antichain.*

Proof. Set $A_1 = \mathrm{MAX}(\mathbf{P})$. Thereafter set $A_{i+1} = \mathrm{MAX}(\mathbf{P}_i)$ where $\mathbf{P}_i$ is the subposet obtained by removing the antichains $A_1, A_2, \ldots, A_i$ from $\mathbf{P}$. □

The first major result in this chapter is an important generalization of Dilworth's chain partitioning theorem due to Greene and Kleitman (1976). The proof we give here is patterned after algorithmic proofs given by Saks (1979) and Perfect (1984). An alternative proof using network flows is given in this volume in chapter 2.

We need some preliminary notation and terminology. Let $\mathbf{P} = (X,P)$ be a poset and k a positive integer. A subset $S \subset X$ is called a *Sperner k-family* if S does not contain a chain of $(k+1)$-elements. The maximum cardinality of a Sperner k-family is denoted $d_k(\mathbf{P})$. When $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ is a family of chains forming a partition of X, we define $e_k(\mathscr{C}) = \sum_{i=1}^{t} \min\{k, |C_i|\}$. If S is any Sperner k-family and $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ is any chain partition of X, we note that $|S \cap C_i| \leqslant \min\{k, |C_i|\}$. Thus $|S| \leqslant e_k(\mathscr{C})$, so that $d_k(\mathbf{P}) \leqslant e_k(\mathscr{C})$. The chain partition $\mathscr{C}$ is said to be *k-saturated* if $d_k(\mathbf{P}) = e_k(\mathscr{C})$.

We also need a preliminary lemma whose elementary proof is omitted. Let $\mathcal{M}(\mathbf{P})$ denote the set of all maximum antichains of $\mathbf{P}$. Define a partial order on $\mathcal{M}(\mathbf{P})$ by $A \leqslant B \iff$ for every $a \in A$, there exists $b \in B$ with $a \leqslant b$.

Lemma 2.3. *The set $\mathcal{M}(\mathbf{P})$ of maximum antichains of a poset $\mathbf{P} = (X, P)$ has a unique greatest element.*

With this background, here is the Greene–Kleitman theorem.

Theorem 2.4. *Let $\mathbf{P}$ be a poset and k a positive integer. Then there exists a chain partition $\mathscr{C}$ of $\mathbf{P}$ which is simultaneously k-saturated and $(k+1)$-saturated, i.e., $d_k(\mathbf{P}) = e_k(\mathscr{C})$ and $d_{k+1}(\mathbf{P}) = e_{k+1}(\mathscr{C})$.*

Proof. We first show that $d_1(\mathbf{P} \times \mathbf{k}) = d_k(\mathbf{P})$ for every $k \geqslant 1$. Let A be a maximum antichain in $\mathbf{P} \times \mathbf{k}$, and let $A_i = \{x \in X\colon (x, i) \in A\}$. Then each A_i is an antichain in $\mathbf{P}$, so the set $S = A_1 \cup A_2 \cup \cdots \cup A_k$ is a Sperner k-family. Furthermore, $|S| = |A|$ since $A_i \cap A_j = \emptyset$ when $i \neq j$. Thus $d_1(\mathbf{P} \times \mathbf{k}) \leqslant d_k(\mathbf{P})$.

Conversely, let S be a maximum Sperner k-family in $\mathbf{P}$. Partition S into k antichains by setting $A_1 = \mathrm{MAX}(S)$ and $A_{i+1} = \mathrm{MAX}(S - (A_1 \cup A_2 \cup \cdots \cup A_i))$. Then $A = \{(a, i)\colon a \in A_i\}$ is an antichain in $\mathbf{P} \times \mathbf{k}$ with $|A| = |S|$. This shows $d_1(\mathbf{P} \times \mathbf{k}) \geqslant d_k(\mathbf{P})$. Thus $d_1(\mathbf{P} \times \mathbf{k}) = d_k(\mathbf{P})$.

For the remainder of the proof, we fix a positive integer k. Then we make several definitions concerning chain partitions of $\mathbf{P} \times (\mathbf{k+1})$. When $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ is a chain partition of $\mathbf{P} \times (\mathbf{k+1})$, we let $M(\mathscr{C}) = \{\mathrm{MAX}(C_i)\colon 1 \leqslant i \leqslant t\}$. We say *$y$ covers x* in $\mathscr{C}$ if there is some $C_i \in \mathscr{C}$ so that y covers x in the chain C_i. When $S \subset \mathbf{P} \times (\mathbf{k+1})$, the set $\{x \in X\colon (x, i) \in S$ for some $i\}$ is called the *projection* of S on $\mathbf{P}$. For each i, the subset $S \cap (\mathbf{P} \times \{i\})$ is called *level i* of S. The projection on $\mathbf{P}$ of level i of $M(\mathscr{C})$ is denoted by $M_i(\mathscr{C})$.

A chain partition $\mathscr{C}$ of $\mathbf{P} \times (\mathbf{k+1})$ is *special* if the following two conditions hold:

(i) $M_0(\mathscr{C}) \supset M_1(\mathscr{C}) \supset M_2(\mathscr{C}) \supset \cdots \supset M_{k-1}(\mathscr{C})$;

(ii) If $x \in M_k(\mathscr{C}) - M_{k-1}(\mathscr{C})$, then (x, k) covers $(x, k-1)$ in $\mathscr{C}$.

A special chain partition of $\mathbf{P} \times \mathbf{k+1}$ is *very special* if it also satisfies the following two conditions:

(iii) Exactly $d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$ of the chains in $\mathscr{C}$ are subsets of level 0; and

(iv) $|\mathscr{C}| = d_1(\mathbf{P} \times (\mathbf{k+1}))$.

When $\mathscr{C}$ is special, it follows from the second condition in this definition that $M_k(\mathscr{C}) = N_1(\mathscr{C}) \cup N_2(\mathscr{C})$ where $N_1(\mathscr{C}) = M_k(\mathscr{C}) \cap M_{k-1}(\mathscr{C})$. If $x \in N_2(\mathscr{C})$, then (x, k) covers $(x, k-1)$ in $\mathscr{C}$.

We now show that the theorem follows whenever $\mathbf{P} \times (\mathbf{k+1})$ has a very special chain partition. To see this, let $\mathscr{C}_{k+1} = \{C_1, C_2, \ldots, C_t\}$ be a very special chain partition of $\mathbf{P} \times (\mathbf{k+1})$ where $t = d_1(\mathbf{P} \times (\mathbf{k+1}))$. Set $s = d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$. We assume that $C_1, C_2, \ldots, C_s$ are subsets of level 0. For each $j = 1, 2, \ldots, t$, let $D_j = \{(x, i)\colon (x, i+1) \in C_j\}$. Of course, $D_1, D_2, \ldots, D_s$ are all empty. Let $\mathscr{C}_k$ be the collection of all nonempty D_j's. Then $\mathscr{C}_k$ is a chain partition of $\mathbf{P} \times \mathbf{k}$ and $|\mathscr{C}_k| \leqslant t - s = d_k(\mathbf{P}) = d_1(\mathbf{P} \times \mathbf{k})$. Thus $|\mathscr{C}_k| = d_k(\mathbf{P})$. Furthermore, it is easy to see that $\mathscr{C}_k$ is a special chain partition of $\mathbf{P} \times \mathbf{k}$.

Now level k in $\mathbf{P} \times (\mathbf{k}+\mathbf{1})$ forms a copy of $\mathbf{P}$ and the $|M_k(\mathscr{C}_{k+1})|$ chains in $\mathscr{C}_{k+1}$ which intersect level k determine a chain partition of $\mathbf{P}$ which we denote by $\mathscr{C}$. We show that $\mathscr{C}$ is both k-saturated and $(k+1)$-saturated.

Now let $j \in \{k, k+1\}$. Then

$$\begin{aligned} d_j(\mathbf{P}) = e_1(\mathscr{C}_j) = |M(\mathscr{C}_j)| &= \sum_{i=0}^{j-1} |M_i(\mathscr{C}_j)| \\ &\geqslant (j-1)|M_{j-2}(\mathscr{C}_j)| + |M_{j-1}(\mathscr{C}_j)| \\ &= (j-1)|M_{j-2}(\mathscr{C}_j)| + |M_{j-1}(\mathscr{C}_j) \cap M_{j-2}(\mathscr{C}_j)| + |N_2(\mathscr{C}_j)| \\ &\geqslant j|M_{j-1}(\mathscr{C}_j) \cap M_{j-2}(\mathscr{C}_j)| + |N_2(\mathscr{C}_j)| \\ &= j|N_1(\mathscr{C}_j)| + |N_2(\mathscr{C}_j)| \\ &\geqslant \sum_{E \in \mathscr{E}} \min\{j, |E|\} \\ &= e_j(\mathscr{E}) \geqslant d_j(\mathbf{P}). \end{aligned}$$

Thus $\mathscr{E}$ is both k-saturated and $(k+1)$-saturated as claimed. To complete the proof, we need only show the existence of a very special chain partition of $\mathbf{P} \times (\mathbf{k}+\mathbf{1})$. Set $t = d_{k+1}(\mathbf{P}) = d_1(\mathbf{P} \times (\mathbf{k}+\mathbf{1}))$. Of all partitions of $\mathbf{P}(k+1)$ into t chains, choose one having as many chains as possible as subsets of level 0. Call this partition $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ and label the chains in $\mathscr{C}$ so that $C_1, C_2, \ldots, C_s$ are subsets of level 0 but $C_{s+1}, C_{s+2}, \ldots, C_t$ are not. Since the last $t-s$ chains in $\mathscr{C}$ cover a copy of $\mathbf{P} \times \mathbf{k}$, we know $t - s \geqslant d_1(\mathbf{P} \times \mathbf{k}) = d_k(\mathbf{P})$, so $s \leqslant d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$. We show that $s = d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$. Suppose to the contrary that $s < d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$.

Let $\mathbf{Q} = \mathbf{P} \times (\mathbf{k}+\mathbf{1}) - (C_1 \cup C_2 \cup \cdots \cup C_s)$. Clearly, the width of $\mathbf{Q}$ is $t-s$. Let A be the unique greatest element in the poset $\mathscr{M}(\mathbf{Q})$ of maximum antichains in $\mathbf{Q}$. A contains at least $d_{k+1}(\mathbf{P}) - d_k(\mathbf{P}) - s$ elements from level 0 since the width of the top k levels is only $d_k(\mathbf{P})$. Choose an element $a_0 \in A$ which comes from level 0. Without loss of generality $a_0 \in C_{s+1}$. Let $\mathbf{Q}' = \mathbf{Q} - \{c \in C_{s+1} \colon c \leqslant a_0\}$.

We claim that the width of $\mathbf{Q}'$ is less than $t-s$, for if $\mathbf{Q}'$ contains a $(t-s)$-element antichain B, then B contains an element b with $a_0 < b$ in $\mathbf{Q} \times (\mathbf{k}+\mathbf{1})$. This contradicts our choice of A. It follows that we can partition Y' into $t-s-1$ chains which together with $C_1, C_2, \ldots, C_s$ and $\{c \in C_{s+1} \colon c \leqslant a_0\}$ form a partition of $\mathbf{P} \times (\mathbf{k}+\mathbf{1})$ into t chains. In this partition, there are $s+1$ chains which are subsets of level 0. The contradiction shows $s = d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$.

We now proceed to transform $\mathscr{C}$ into a very special partition by a series of operations called *insertions* and *switches*. At this moment $\mathscr{C}$ satisfies properties (iii) and (iv), and both operations preserve these properties.

We first perform a series of insertions. Choose points (x, i), (y, j) so that (x, i) covers (y, j) in $\mathscr{C}$ and $i > j$. If $i \neq j+1$ or $x \neq y$, remove $(y, j+1)$ from the chain to which it currently belongs and insert it in the chain containing (x, i) and (y, j). Repeat until no further insertions are possible.

Next, we perform a series of switches. For an integer $j \geqslant 1$, locate a point $(x,j) \in M(\mathscr{C})$ so that either: (1) $j < k$ and $(x, j-1) \notin M(\mathscr{C})$; or (2) $j = k$ and (x,j) does not cover $(x, j-1)$ in $\mathscr{C}$, and $(x, j-1) \notin M(\mathscr{C})$.

Let (y,i) be the point covering $(x, j-1)$ and let C be the chain containing $(y, i+1)$. Let C'' consist of those points in $\mathscr{C}$ which are less than $(y, i+1)$ and let $C'' = C - C'$. Then let D be the chain containing (x,j) and set $D' = D \cup C''$. Replace C and D in $\mathscr{C}$ by C' and D'. Repeat until no further switches are possible.

It is obvious that the series of insertions must stop, but it takes a moment's reflection to see that this is also true for the series of switches. For $j = 1, 2, \ldots, k$, let v_j count the number of points $x \in X$ for which (x,j) covers (y,i) in $\mathscr{C}$ and $(x, j-1)$ covers $(y, i-1)$ in $\mathscr{C}$. Each time we perform a switch, the vector $(v_1, v_2, \ldots, v_k)$ increases lexicographically. Since $v_j \leqslant |X|$ for each j, the procedure stops. $\square$

This theorem has many significant applications. The following corollary follows easily, but we know of no simple proof avoiding the use of Theorem 2.4.

Corollary 2.5. *Let* $\mathbf{P}$ *be a finite poset. Then for each* $k \geqslant 1$, $d_k(\mathbf{P}) - d_{k-1}(\mathbf{P}) \geqslant d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$.

3. Kierstead's chain partitioning theorem

In this section we outline the proof of a theorem of Kierstead (1981) which asserts that for each $n \geqslant 1$, there is a $t = t(n)$ for which there exists an on-line algorithm which will partition any poset $\mathbf{P}$ of width at most n into t chains. By an on-line partition, we mean that the poset and the partition are constructed one point at a time. An adversary (infinitely clever) constructs the poset and we must devise the partition. At each round, the adversary presents the new point and describes its comparabilities and incomparabilities to all preceding points. We must then add the new point to one of the sets making up the partition. Both players' moves are permanent.

As a warm-up, we first present the on-line version of the dual to Dilworth's theorem. The result is an unpublished theorem of Schmerl, although a short proof is given in Kierstead (1986).

Theorem 3.1. *For each* $n \geqslant 1$, *there exists an algorithm which will construct an on-line partition of a poset of height at most* n *into* $n(n+1)/2$ *antichains.*

Proof. When the new point x is added to the poset, let $r = r(x)$ be the maximum number of points in a chain having x as least element, and let $s = s(x)$ be the maximum number of points in a chain having x as greatest element. Assign x to the set $A(r,s)$. Clearly, each $A(r,s)$ is an antichain. Since $r + s - 1 \leqslant n$, there are $n(n+1)/2$ such sets. $\square$

Szemerédi produced a simple argument to show that Theorem 3.1 is best possible, and we invite the reader to reconstruct his proof. Full details are given in

Kierstead (1986). As a first step, show that there exists a strategy for constructing a poset **P** of height at most n which will force any opponent producing an on-line partition into antichains to use at least $n(n+1)/2$ antichains in covering **P** and at least n antichains in covering MAX(**P**).

Here is Kierstead's on-line chain partitioning theorem (Kierstead 1981).

Theorem 3.2. *For each $n \geqslant 1$, there exists an algorithm which will construct an on-line partition of a poset of width at most n into $(5^n - 1)/4$ chains.*

Proof. The argument proceeds by induction on n with the case $n = 1$ being trivial. The heart of the argument is the case $n = 2$ where we have to partition a width-2 poset into 6 chains.

We first construct a greedy chain C_1. As a new point enters the poset, we insert it in C_1 whenever it is comparable to all other points previously placed in C_1. Thus for every $x \in X - C_1$, there is a nonempty set $I(x)$ of points from C_1 which are incomparable to x. Although $I(x)$ may grow with time, it is always a set of consecutive points from C_1. When $x, y \in X - C_1$, we write $I(x) < I(y)$ when $u < v$ for every $u \in I(x)$ and every $v \in I(y)$. Note that if x and y are incomparable points in $X - C_1$, then the following condition holds:

(K). When the latter of x and y enters, either

$$I(x) < I(y) \text{ or } I(y) < I(x).$$

In fact, when $n = 2$, the qualifying phrase "when the latter of x and y enters" can be dropped since $I(x) \cap I(y) = \emptyset$ whenever $x \parallel y$. Regardless, we choose the weaker statement since it is crucial to the inductive step.

We define a partial order, called the $*$-order, on $X - C_1$ as follows. When the new point x enters, we set $x * y$ if

(1) $x < y$ in $\mathbf{P} - C_1$, or

(2) $x \parallel y$ and $I(x) < I(y)$.

Similarly, we set $y * x$ if

(3) $y < x$ in $\mathbf{P} - C_1$, or

(4) $x \parallel y$ and $I(y) < I(x)$.

With this definition, it is straightforward to verify that $(X - C_1, *)$ is a chain, i.e., $*$ is a linear extension of the original partial order on $X - C_1$. Next, we define an equivalence relation on $\mathbf{P} - C_1$. Just as is the case with the $*$-order, the definition of this equivalence relation is on-line. The relation will satisfy:

(a) each equivalence class is a set of consecutive elements of $X - C_1$ in the $*$-order, and

(b) if x and y are consecutive elements belonging to the same equivalence class, then $I(x) \cap I(y) \neq \emptyset$.

When a new point x enters $X - C_1$, we put x in the same equivalence class as y if $x <: y$ in $*$ and $I(x) \cap I(y) \neq \emptyset$. If no such y exists, we put x in the same class as z if $z <: x$ in $*$ and $I(x) \cap I(z) \neq \emptyset$. If neither of these results in the assignment of x to an existing class, start a new equivalence class whose only element (at this moment) is x.